كلية الهندسة – جامعة مصراتة

القسم العام

رياضة "2"

الشيت الثالث

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⑦ التكامل بالتعويض:

قاعدة: تعتبر طريقة التعويض إحدى الطرق المهمة في إيجاد التكامل.
يمكن استخدام التعويض في حل الطرق السابقة، ولكن أحياناً نفشل الطرق ونلجأ لطريقة التعويض.

أي أن:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u).$$

Q_7: Find the following int.

① $\int \frac{x}{\sqrt{1+6x}}\, dx.$ نفرض u تحت الجذر

$u = 1+6x \longrightarrow x = \frac{1}{6}(u-1) \xrightarrow{\text{تفاضل}} dx = \frac{1}{6}\,du$

$I = \int \frac{\frac{1}{6}(u-1)\cdot\frac{1}{6}du}{\sqrt{u}} = \frac{1}{36}\int \frac{u-1}{\sqrt{u}}\, du$

$= \frac{1}{36}\int u^{1/2} - u^{-1/2}\, du = \frac{1}{36}\left[\frac{u^{3/2}}{\frac{3}{2}} - \frac{u^{1/2}}{\frac{1}{2}}\right] + c.$

$= \frac{1}{36}\left[\frac{2}{3}(1+6x)^{3/2} - 2(1+6x)^{1/2}\right] + c.$

② $\int x^2\sqrt{x+a}\, dx.$

$u = x+a \longrightarrow x = u-a \xrightarrow{\text{تفاضل}} \boxed{dx = du}$

$\int (u-a)^2\sqrt{u}\, du = \int (u^2 - 2au + a^2)\, u^{1/2}\, du.$

$$\int (u^{7/2} - 2au^{3/2} + a^2 u^{1/2})\, du = \frac{u^{7/2}}{7/2} - 2a\frac{u^{5/2}}{5/2} + a^2\frac{u^{3/2}}{3/2} + c$$

$$\frac{2}{7}u^{7/2} + \frac{4a}{5}u^{5/2} + \frac{2a^2}{3}u^{3/2} + c.$$

$$I = \frac{2}{7}(x+a)^{7/2} + \frac{4a}{5}(x+a)^{5/2} + \frac{2a^2}{3}(x+a)^{3/2} + c.$$

③ $\int (x+1)^{10}(x+2)\, dx.$

$u = x+1 \rightarrow x = u-1 \rightarrow dx = du.$

$$\int u^{10}(u-1+2)\, du = \int u^{10}(u+1)\, du.$$

$$\int (u^{11} + u^{10})\, du = \frac{u^{12}}{12} + \frac{u^{11}}{11} + c.$$

$$I = \frac{1}{12}(x+1)^{12} + \frac{1}{11}(x+1)^{11} + c.$$

④ $\int x^5\sqrt{x^3-1}\, dx.$

$u = x^3 - 1 \rightarrow x^3 = u+1 \xrightarrow{\text{تفاضل}} 3x^2 dx = du$

$x^2 dx = \frac{1}{3}du.$

$$I = \int x^3 \cdot x^2\sqrt{x^3-1}\, dx = \int (u+1)\sqrt{u}\, \frac{1}{3}du$$

$$\frac{1}{3}\int (u^{3/2} + u^{1/2})\, du = \frac{1}{3}\left[\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right] + c.$$

$$I = \frac{1}{3}\left[\frac{2}{5}(x^3-1)^{5/2} + \frac{2}{3}(x^3-1)^{3/2}\right] + c.$$

⑤ $\displaystyle\int \frac{e^{2x}}{e^x+1}\,dx.$

$u = e^x+1 \rightarrow e^x = u-1 \xrightarrow{\text{تفاضل}} e^x dx = du.$

$$\int \frac{(u-1)\,du}{u} = \int \left(1-\frac{1}{u}\right) du$$

$$I = u - \ln|u| + c = (e^x+1) \ln|e^x+1| + c.$$

⑥ $\displaystyle\int_0^4 x\sqrt{2x+1}\,dx.$

$u = 2x+1 \rightarrow x = \frac{1}{2}(u-1) \rightarrow dx = \frac{1}{2}du.$

$$\int_0^4 \frac{1}{2}(u-1)\sqrt{u} \times \frac{1}{2}du = \int_0^4 \frac{1}{4}\left(u^{3/2} - u^{1/2}\right) du.$$

$$= \frac{1}{4}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + c = \frac{1}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + c.$$

$$I = \frac{1}{4}\left[\frac{2}{5}(2x+1)^{5/2} - \frac{2}{3}(2x+1)^{3/2}\right] + c.$$

⑧ التكامل بالتجزيء :

تعتمد هذه الطريقة على قانون حاصل ضرب دالتين .

$$d(u.v) = u\,dv + v\,du$$

$$u.v = \int u.dv + \int v.du \rightarrow \boxed{\int u\,dv = u.v - \int v\,du}$$

* يستخدم التكامل بالتجزيء في الحالات الآتية :

① اذا وجد حاصل ضرب متعددة حدود (x) دالة أسية أو مثلثية أو زائدية ،
② إذا وجد حاصل ضرب دالة مثلثية (x) أسية أو زائدية .
③ إذا وجدت دالة لوغاريتمية .
④ إذا وجدت دالة عكسية أو دالة زائدية عكسية .
⑤ $\int \sec^n x\,dx$ حيث (n) عدد فردي .

* تكون أسبقية لاختيار (u) كما يلي :

① دالة لوغارتيمية .
② دالة مثلثية عكسية أو زائدية عكسية .
③ متعددة الحدود .
④ دالة أسية أو مثلثية أو زائدية .

Q8: Find the following int.

① $\int x^2 e^x\,dx$

نفرض $u = x^2 \xrightarrow{\text{تفاضل}} du = 2x\,dx.$

$dv = e^x dx \xrightarrow{\text{تكامل}} v = e^x.$

$$I = x^2 e^x - \underline{\int 2x\, e^x\, dx}.$$

نعاود الفرضية مرة أخرى .

$u = 2x \rightarrow du = 2\,dx$

$dv = e^x\,dx \longrightarrow v = e^x.$

تفاضل		تكامل
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

$$I = x^2 e^x - \underline{\left[2xe^x - \int 2e^x dx\right]}$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

② $\int x^3 e^{x^2} dx. \quad = \int x^2\; x e^{x^2} dx$

$u = x^2 \longrightarrow du = 2x\,dx$

$dv = x e^{x^2} \longrightarrow \frac{1}{2} e^{x^2}.$

$$I = \frac{1}{2}x^2 e^{x^2} - \int \frac{1}{2} e^{x^2}\, 2x\, dx.$$

$$= \frac{1}{2}x^2 e^{x^2} - \int x e^{x^2} dx.$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C.$$

③ $\int x \sec^2 x\; dx.$

$u = x \longrightarrow du = dx$

$dv = \sec^2 x\; dx \longrightarrow v = \tan x$

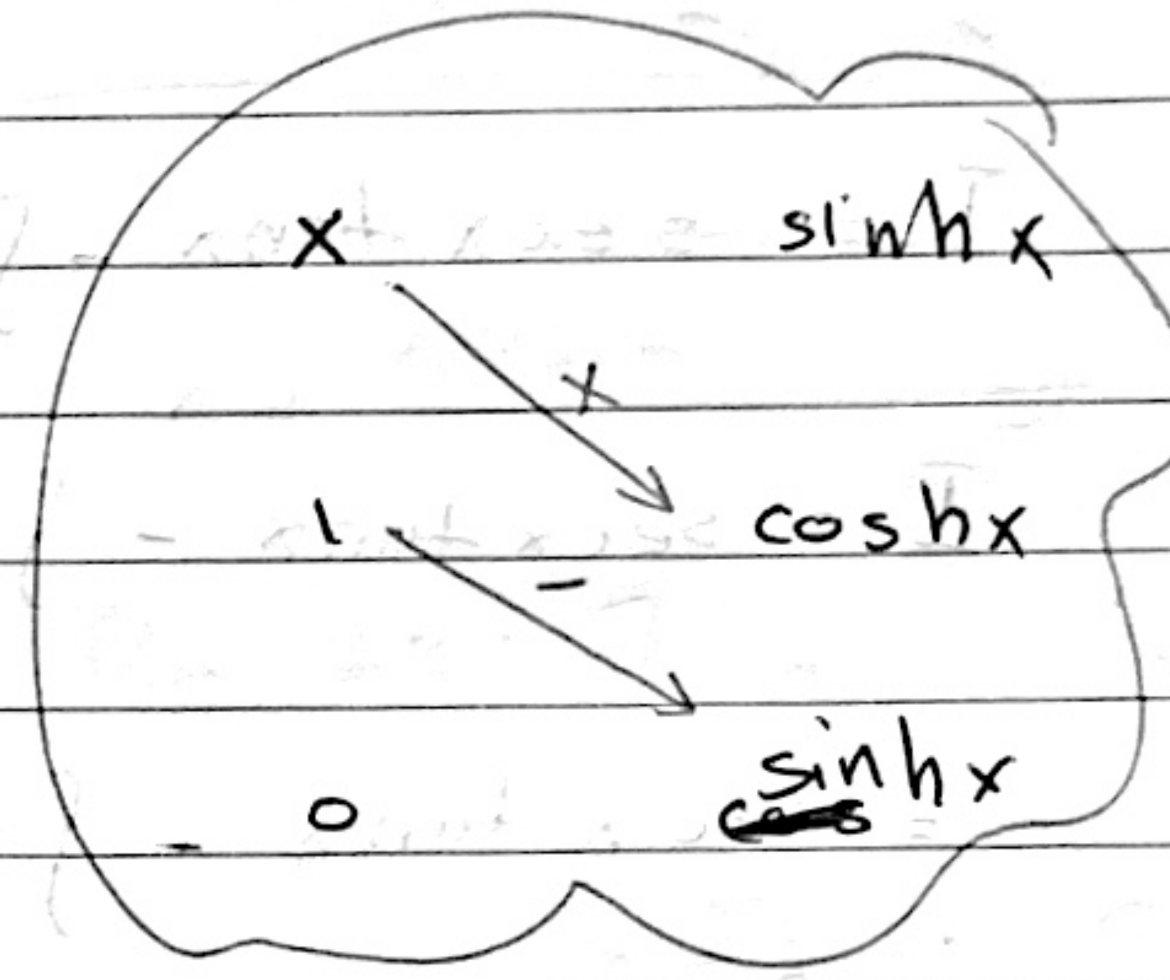
$$I = x\tan x - \int \tan x \, dx$$

$$= x\tan x - \ln|\sec x| + C.$$

④ $\int x \sinh x \, dx$

$u = x \longrightarrow du = dx.$

$dv = \sinh x \, dx \longrightarrow v = \cosh x.$



$$I = x\cosh x - \int \cosh x \, dx$$

$$= x\cosh x - \sinh x + C.$$

⑤ $\int e^x \cos 2x \, dx.$

$u = e^x \longrightarrow du = e^x dx$

$dv = \cos 2x \, dx \longrightarrow \frac{1}{2}\sin 2x.$

$$= \frac{1}{2}e^x \sin 2x - \int \frac{1}{2}\sin 2x \, e^x dx.$$

تعيد الفرضية

$u = e^x \longrightarrow du = e^x dx.$

$dv = \sin 2x \, dx \longrightarrow \frac{-1}{2}\cos 2x.$

$$I = \frac{1}{2}e^x \sin 2x - \left[\frac{1}{2}\left[\frac{-1}{2}e^x \cos 2x - \int \frac{-1}{2} \underbrace{\cos 2x \, e^x}_{I}\right]\right]$$

عاود نفس التكامل الاول

$$I = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}I$$

$$\frac{5}{4}I = \frac{1}{4}e^x(\sin 2x + \cos 2x) \Longrightarrow I = \frac{1}{5}e^x(\sin 2x + \cos 2x) + C.$$

$\textcircled{6} \int \sec^3 x \quad dx = \int \sec^2 x \cdot \sec x \; dx.$

$u = \sec x \rightarrow du = \sec x \tan x \cdot dx.$

$dv = \sec^2 x \, dx \longrightarrow v = \tan x.$

$I = \sec x \tan x - \int \tan^2 x \sec x \; dx.$

$I = \sec x \tan x - \int (\sec^2 x - 1) \; \sec \, dx.$

$= \sec x \tan x - \int (\sec^3 x \; - \sec x) \, dx$

$I = \sec x \tan x - I - \ln|\sec x + \tan x|$

$2I = \sec x \tan x - \ln|\sec x + \tan x|$

$I = \frac{1}{2} (\sec x \tan x - \ln|\sec x + \tan x| + c$

⑦ $\int \ln x \, dx$

$u = \ln x \longrightarrow du = \frac{1}{x} dx.$

$dv = dx \longrightarrow v = x.$

$= x \ln x - \int x \cdot \frac{1}{x} dx.$

$= x \ln x - x + C.$

⑧ $\int \sqrt{x} (\ln \sqrt{x})^2 dx.$

$\int x^{1/2} (\ln x^{1/2})^2 dx \longrightarrow \int x^{1/2} (\frac{1}{2} \ln x)^2 dx.$

$\frac{1}{4} \int x^{1/2} (\ln x)^2 dx.$

$u = (\ln x)^2 \longrightarrow du = 2(\ln x) \cdot \frac{1}{x} dx.$

$dv = x^{1/2} dx \longrightarrow v = \frac{2}{3} x^{3/2},$

$I = \frac{1}{4} \left[\frac{2}{3} x^{3/2} (\ln x)^2 - \frac{4}{3} \int (\ln x) \cdot x^{1/2} dx \right]$

$I = \frac{1}{6} x^{3/2} (\ln x)^2 - \frac{1}{3} \int (\ln x) x^{1/2} dx.$

$u = \ln x \longrightarrow du = \frac{1}{x} dx.$

$dv = x^{1/2} dx \longrightarrow v = \frac{2}{3} x^{2/3}.$

$I = \frac{1}{6} x^{3/2} (\ln x)^2 - \frac{1}{3} \left[\frac{2}{3} x^{3/2} \cdot \ln x - \frac{2}{3} \int x^{1/2} dx \right]$

$I = \frac{1}{6} x^{3/2} (\ln x)^2 - \frac{2}{9} x^{3/2} \ln x - \frac{2}{9} \left(\frac{2}{3} x^{3/2} \right) + C.$

(9) $\int \ln(x^3 + x^2)\,dx$

$$\int \ln\left[x^2[x+1]\right] dx \longrightarrow \int (\ln x^2 + \ln(x+1))\,dx$$

$$u = \ln(x+1) \longrightarrow du = \frac{1}{x+1}\,dx$$

$$dv = dx \longrightarrow v = x$$

$$I = 2(x\ln|x| - x) + (x+1)\ln|x+1| - (x+1) + C$$

(10) $\int_0^1 3x^2 e^{2x}\,dx \longrightarrow 3\int_0^1 x^2 e^{2x}\,dx$

u		dv
x^2	$+$	e^{2x}
$2x$	$-$	$\frac{1}{2}e^{2x}$
2	$+$	$\frac{1}{4}e^{2x}$
0		$\frac{1}{8}e^{2x}$

$$I = 3\left(\frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}\right)\Big|_0^1$$

$$I = \frac{3}{2}x^2 e^{2x} - \frac{3}{2}x e^{2x} - \frac{3}{4}e^{2x}\Big|_0^1$$

$$I = \frac{3}{2}e^2 - \frac{3}{2}e^2 + \frac{3}{4}e^2 - \frac{3}{4}$$

$$I = \frac{3}{4}e^2 - \frac{3}{4}.$$

⑨ التكاملات المثلثية :

① $\int \sin^m x \cos^n x \, dx$.

if m odd $\rightarrow$ نفرض $u = \cos x \rightarrow du = -\sin x \, dx$.

if n odd $\rightarrow$ نفرض $u = \sin x \rightarrow du = \cos x \, dx$.

if m, n even $\rightarrow$ نفرض $\rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

② $\int \tan^m x \sec^n x \, dx$.

if m even or odd $\rightarrow$ نفرض $\tan^2 x = \sec^2 x - 1$

if n even $\rightarrow$ نفرض $u = \tan x \rightarrow du = \sec^2 x \, dx$.

if m, n odd $\rightarrow$ نفرض $u = \sec x \rightarrow du = \sec x \tan x \, dx$.

③ $\int \cot^m x \csc^n x \, dx$.

if m even or odd $\rightarrow$ نفرض $\cot^2 x = \csc^2 x - 1$

if n even. $\rightarrow$ نفرض $u = \cot x \rightarrow du = -\csc^2 x \, dx$.

if m, n odd. $\rightarrow$ نفرض $u = \csc x \rightarrow du = -\csc x \cot x \, dx$.

Q9: Find the following int.

① $\int \sin^3 x \cos^2 x \, dx.$

$I = \int \sin^2 x \sin x \cos^2 x \, dx.$

$I = \int (1-\cos^2 x) \sin x \cos^2 x \, dx. \longrightarrow \int (\cos^2 x - \cos^4 x) \sin x \, dx.$

فرض $u = \cos x \longrightarrow du = -\sin x \, dx,$

$I = \int (u^2 - u^4) \cdot (-du) = -\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + c$

$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c.$

② $\int \sin^4 x \cos^3 x \, dx.$

$I = \int \sin^4 x \cos^2 x \cos x \, dx.$

$= \int \sin^4 x (1-\sin^2 x) \cos x \, dx.$

$\int (\sin^4 x - \sin^6 x) \cos x \, dx.$

$u = \sin x \longrightarrow du = \cos x \, dx$

$\int (u^4 - u^6) \, du = \frac{u^5}{5} - \frac{u^6}{6} + c.$

$= \frac{(\sin x)^5}{5} - \frac{(\sin x)^6}{6} + c.$

③ $\int \sin^2 x \cos^2 x \, dx.$

$I = \int \frac{1}{2}(1-\cos 2x) \cdot \frac{1}{2}(1+\cos 2x)\, dx.$

$\frac{1}{4}\int (1-\cos^2 2x)\, dx = \frac{1}{4}\int \sin^2 2x \, dx.$

$\frac{1}{4} \cdot \frac{1}{2}\int (1-\cos 4x)\, dx$

$\frac{1}{8}\left[x - \frac{1}{4}\cos 4x\right] + C$

④ $\int \tan^2 x \sec x \, dx.$

$I = \int (\sec^2 x - 1) \cdot \sec x \, dx. = \int (\underline{\sec^3 x} - \sec x)\, dx.$ ← أخذنا اجابته

$= \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] - \ln|\sec x + \tan x| + C.$

⑤ $\int \sec^4 x \tan^2 x \, dx.$

$I = \int \sec^2 x \cdot \sec^2 x \tan^2 x \, dx. = \int (\tan^2 x + 1) \tan^2 x \sec^2 x \, dx.$

$\int (\tan^2 x + \tan^4 x) \cdot \sec^2 x \, dx.$

$u = \tan x \longrightarrow du = \sec^2 x \, dx$

$= \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$

⑥ $\int \tan^3 x \sec^3 x \, dx$.

$I = \int \tan^2 x . \tan x . \sec^3 x \, dx = \int (\sec^2 x - 1) \tan x . \sec^3 x \, dx$

$\int (\sec^4 x - \sec^2 x) . \sec x \tan x \, dx$.

$u = \sec x \longrightarrow du = \sec x \tan x . dx$.

$\int (u^4 - u^2) \, du \longrightarrow = \frac{u^5}{5} - \frac{u^3}{3} + C$.

$\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$.

⑦ $\int \cot^3 x \, dx$

$I = \int \cot^2 x \cot x \, dx = \int (\csc^2 x - 1) \cot x \, dx$.

$\int (\csc^2 x \cot x - \cot x) \, dx = -\frac{\cot^2 x}{2} - \ln|\sin x| + C$.

⑧ $\int \frac{dx}{\sin^4 x}$

$\int \csc^4 x \, dx = \int \csc^2 x . \csc^2 x \, dx$.

$\int (1 + \cot^2 x) . \csc^2 x \, dx$.

$\int \csc^2 x + \cot^2 x . \csc^2 x \, dx$.

$I = -\cot x - \frac{\cot^3 x}{3} + C$.

⑨ $\int x \tan^3 5x^2 \, dx.$

$= \int x \tan 5x^2 \cdot \tan^2 5x^2 \cdot dx.$

$\int x \tan 5x^2 \, (\sec^2 5x^2 - 1) \, dx$

$\int (x \tan 5x^2 \sec^2 5x^2 - x \tan 5x^2) \, dx$

$I = \frac{1}{20} \tan^2 5x^2 - \frac{1}{10} \ln|\sec 5x^2| + c.$