



كلية الهندسة - جامعة مصراتة

القسم العام

رياضة "2"

الشيت الخامس

أ. عُمر علي بن ساسي

* تكامل الدوال القياسية (الأسور الجزئية):

نستخدم هذه الطريقة عندما يكون التكامل عبارة عن $\frac{g(x)}{p(x)}$ بحيث أن البسط والمقام عبارة عن مخدات حدود.

وتكونا درجة البسط $>$ من درجة المقام، البسط $\neq$ تقاضل المقام.

للمريقة حل هذه التاملات تكون كالتالي:

① إذا كان المقام عبارة عن حوامل ضرب عوامل خطية، أي أن

$$\frac{g(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}.$$

$$\frac{g(x)}{(x-a)^2(x-b)} = \frac{A}{(x-a)^2} + \frac{B}{x-a} + \frac{C}{x-b}.$$

② إذا كان المقام عبارة عن حوامل ضرب عوامل تربيعية، أي أن

$$\frac{g(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)} = \frac{Ax+B}{a_1x^2+b_1x+c_1} + \frac{Dx+E}{a_2x^2+b_2x+c_2}.$$

$$\frac{g(x)}{(a_1x^2+c_1)^2(a_2x^2+b_2x+c_2)} = \frac{Ax+B}{(a_1x^2+c_1)^2} + \frac{Cx+D}{a_1x^2+c_1} + \frac{Ex+F}{a_2x^2+b_2x+c_2}.$$

Q₁₄: Find the following int.

$$\textcircled{1} \int \frac{dx}{x^2 - x} \rightarrow \int \frac{1}{x(x-1)} dx.$$

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \rightarrow \frac{A(x-1) + Bx}{x(x-1)}.$$

$$\text{at } x=0 \quad 1 = -A \rightarrow \boxed{A = -1}$$

$$\text{at } x=1 \quad 1 = B \rightarrow \boxed{B = 1}$$

$$I = \int \frac{-1}{x} dx + \int \frac{1}{x-1} dx.$$

$$I = -\ln x + \ln|x-1| + C.$$

$$\textcircled{2} \int \frac{dx}{1-x^2} \rightarrow \int \frac{1}{(1-x)(1+x)} dx.$$

$$\frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x} \rightarrow \frac{A(1+x) + B(1-x)}{(1-x)(1+x)}.$$

$$1 = 2A \rightarrow A = \frac{1}{2}.$$

$$1 = -2B \rightarrow B = -\frac{1}{2}.$$

$$I = \frac{1}{2} \int \frac{1}{1-x} dx + \frac{1}{2} \int \frac{1}{1+x} dx$$

$$I = \frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x| + C.$$

$$\textcircled{3} \int \frac{x}{x^2+4x+3} dx \rightarrow \int \frac{x}{(x+1)(x+3)} dx.$$

$$\frac{x}{(x+1)(x+3)} = \frac{A(x+3) + B(x+1)}{(x+1)(x+3)}.$$

$$-1 = 2A \rightarrow \cancel{1} = \cancel{2} \frac{1}{2} A = -\frac{1}{2}$$

$$-3 = -2B \rightarrow B = \frac{3}{2}$$

$$\int \frac{x}{x^2+4x+3} dx = \frac{-1}{2} \int \frac{1}{x+1} dx + \frac{3}{2} \int \frac{1}{x+3} dx.$$

$$-\frac{1}{2} (\ln|x+1|) + \frac{3}{2} (\ln|x+3|) + C.$$

$$\textcircled{4} \int \frac{x^3+x^2}{x^2+x-2} dx.$$

$$\begin{array}{r} x^2+x-2 \overline{) x^3+x^2} \\ \underline{-x^3+x^2-2x} \end{array}$$

$$\int x + \frac{2x}{x^2+x-2} dx.$$

$$\frac{2x}{(x+2)(x-1)} = \frac{A(x-1) + B(x+2)}{(x+2)(x-1)}.$$

$$-4 = 3A \rightarrow A = -\frac{4}{3}$$

$$2 = 3B \rightarrow B = \frac{2}{3}$$

$$\int x dx + \frac{4}{3} \int \frac{1}{x+2} dx + \frac{2}{3} \int \frac{1}{x-1} dx.$$

$$I = \frac{1}{2}x^2 + \frac{4}{3}(\ln|x+2|) + \frac{2}{3}\ln|x-1| + C.$$

$$(5) \int \frac{8}{x^4 + 2x^3} dx \rightarrow \int \frac{8}{x^3(x+2)}$$

$$\frac{8}{x^3(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2}$$

$$8 = A(x+2) + B(x)(x+2) + C(x^2)(x+2) + Dx^3$$

$$8 = Ax + 2A + Bx^2 + B2x + Cx^3 + C2x^2 + Dx^3$$

$$8 = (C+D)x^3 + (B+2C)x^2 + (A+2B)x + 2A$$

$$2A = 8 \rightarrow \boxed{A = 4}$$

$$A + 2B = 0 \rightarrow \boxed{B = -2}$$

$$B + 2C = 0 \rightarrow \boxed{C = 1}$$

$$C + D = 0 \rightarrow \boxed{D = -1}$$

$$I = \int \frac{4}{x^3} dx + \int \frac{2}{x^2} dx + \int \frac{1}{x} dx - \int \frac{1}{x+2} dx$$

$$I = -\frac{4}{2}x^{-2} + \frac{2}{x} + \ln x - \ln(x+2) + C$$

$$⑥ \int \frac{x+3}{2x^3-8x} dx = \int \frac{x+3}{2x(x-2)(x+2)} dx.$$

$$x+3 = \frac{A}{2x} + \frac{B}{x+2} + \frac{C}{x-2}.$$

$$x+3 = A(x-2)(x+2) + 2Bx(x+2) + 2C(x-2).$$

$$x+3 = (A+2B)x^2 + (4B+2C)x + (-4A-4B)$$

$$A+2B = 0 \rightarrow A = -2B \rightarrow ①$$

$$4B+2C = 1 \rightarrow ②$$

$$-4A-4C = 3 \rightarrow ③$$

$$8B-4C = 3$$

$$\cancel{8B}$$

$$-8B-4C = -2$$

$$-8C = 1$$

$$\boxed{C = \frac{1}{-8}}$$

بتقس الخطوات فوق
القيم الأخرى.

$$I = \frac{-5}{16} \int \frac{1}{x} dx + \frac{5}{16} \int \frac{1}{x-2} dx - \frac{1}{8} \int \frac{1}{x+2} dx$$

$$I = \frac{-5}{16} \ln x + \frac{5}{16} \ln|x-2| - \frac{1}{8} \ln|x+2| + C.$$

$$\textcircled{7} \int \frac{dt}{t^4 + 4t^2 + 3} = \int \frac{1}{(t^2+1)(t^2+3)} dt.$$

$$\frac{1}{(t^2+1)(t^2+3)} = \frac{A^* + B}{t^2+1} + \frac{ct+D}{t^2+3}.$$

$$1 = (A+c)t^3 + (B+D)t^2 + (3A+c)t + (3B+D).$$

$$A+c=0 \rightarrow \textcircled{1}, \quad B+D=0 \rightarrow \textcircled{2}$$

$$3A+c=0 \rightarrow 2A=0 \rightarrow A=0, \quad c=0.$$

$$3B+D=1 \rightarrow 2B=1 \rightarrow B=\frac{1}{2}, \quad D=-\frac{1}{2}.$$

$$I = \frac{1}{2} \int \frac{1}{t^2+1} dt + \frac{-1}{2} \int \frac{1}{t^2+3} dt.$$

$$I = \frac{1}{2} \tan^{-1} t + \frac{-1}{2\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + C.$$

$$\textcircled{8} \int \frac{x^2+x+2}{(x^2+1)(x-1)} dx.$$

$$\frac{x^2+x+2}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

$$x^2+x+2 = (A+C)x^2 + (B-A)x + (C-B)$$

$$A+C=1 \rightarrow C=1-A$$

$$B-A=1 \rightarrow B=1+A$$

$$C-B=2$$

$$A = -1, B = 0, C = 2.$$

$$I = \int \frac{-x}{x^2+1} dx + \int \frac{2}{x-1} dx.$$

$$I = -\frac{1}{2} \ln(x^2+1) + 2 \ln(x-1) + C.$$

$$\textcircled{9} \int \frac{5x^2+3}{(2x-1)^2(x^2+4)} dx.$$

$$\frac{5x^2+3}{(2x-1)^2(x^2+4)} = \frac{A}{(2x-1)^2} + \frac{B}{2x-1} + \frac{Cx+D}{x^2+4}.$$

$$5x^2+3 = (2B+4C)x^3 + (A-B-4C+4D)x^2 + (8B+C-4D)x + (4A-4B+D)$$

$$2B+4C=0 \rightarrow B=-2C \rightarrow \textcircled{1}$$

$$A-B-4C+4D=5 \rightarrow \textcircled{2}$$

$$8B+C-4D=0 \rightarrow -16C+C-4D=0 \rightarrow D = -\frac{15}{4} \rightarrow \textcircled{3}$$

$$4A-4B+D=3 \rightarrow A = \frac{3}{4} - \frac{17}{16}C \rightarrow \textcircled{4}$$

$$\frac{3}{4} - \frac{17}{16}C + 2C - 4C - 15C = 5,$$

$$C = -\frac{4}{17}, B = \frac{8}{17}, D = \frac{15}{17}, A = 1.$$

$$I = \int \frac{1}{(2x-1)^2} dx + \frac{8}{17} \int \frac{1}{2x-1} dx + \int \frac{-\frac{4}{17}x + \frac{15}{17}}{x^2+4} dx.$$

$$I = \int (2x-1)^{-2} dx + \frac{8}{17} \int \frac{dx}{2x-1} - \frac{4}{17} \int \frac{x}{x^2+4} dx + \frac{15}{17} \int \frac{dx}{x^2+4}$$

$$I = \frac{-1}{2(2x-1)} + \frac{4}{17} \ln|2x-1| - \frac{2}{17} \ln|x^2+4| + \frac{15}{34} \tan^{-1} \frac{x}{2} + C$$

$$(10) \int \frac{dx}{x(\sqrt{x} + \sqrt[4]{x})}$$

$$\text{دېځي } x = u^4, \sqrt[4]{x} = u.$$

$$dx = 4u^3 du, \sqrt{x} = u^2.$$

$$\int \frac{4u^3}{u^5(u+1)} du \rightarrow 4 \int \frac{1}{u^2(u+1)} du.$$

$$\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$$

$$1 = A(u^2+u) + Bu + B + Cu^2$$

$$1 = Cu^2 + Au^2 + (A+B)u + B$$

$$\boxed{B=1}, A+B=0 \rightarrow \boxed{A=-1}$$

$$A+C=0 \rightarrow \boxed{C=1}$$

$$4 \int \frac{1}{u^2(u+1)} du = 4 \left[-\int \frac{1}{u} du + \int \frac{1}{u^2} du + \int \frac{1}{u+1} du \right]$$

$$4 \left[-\ln|u| - \frac{1}{u} + \ln|u+1| \right] + C.$$

$$4 \left[-\ln|\sqrt[4]{x}| - \frac{1}{\sqrt[4]{x}} + \ln|\sqrt[4]{x}+1| \right] + C.$$

$$(11) \int \frac{\sqrt[4]{x}}{\sqrt[3]{x} + \sqrt{x}} dx$$

$$x = u^{12} \rightarrow \sqrt{x} = \sqrt{u^{12}} = u^6, \\ dx = 12u^{11} du. \quad \sqrt[3]{x} = \sqrt[3]{u^{12}} = u^4, \\ \sqrt[4]{x} = \sqrt[4]{u^{12}} = u^3.$$

$$\int \frac{u^3}{u^4 + u^6} 12 \cdot u^{11} du.$$

$$12 \int \frac{u^{10}}{1 + u^2} du.$$

درجة البسط < المقام
فئة مقبولة

$$\begin{array}{r} u^8 + u^6 + u^4 - u^2 + 1 \quad - \sqrt{u^2 + 1} \\ \hline u^2 + 1 \quad u^{10} \\ \hline u^{10} + u^8 \\ \hline u^8 \\ \hline u^8 + u^6 \\ \hline u^6 \\ \hline u^6 + u^4 \\ \hline u^4 \\ \hline u^4 + u^2 \\ \hline u^2 \\ \hline u^2 + 1 \\ \hline 1 \end{array}$$

$$12 \int \frac{u^{10}}{1 + u^2} du = 12 \left[\int u^8 - u^6 + u^4 - u^2 + 1 - \frac{1}{u^2 + 1} du \right]$$

$$= 12 \left[\frac{1}{9} u^9 - \frac{1}{7} u^7 + \frac{1}{5} u^5 - \frac{1}{3} u^3 + u - \tan^{-1} u \right] + C.$$

$$I = \frac{4}{3} x^{\frac{9}{12}} + \frac{12}{7} x^{\frac{7}{12}} + \frac{12}{5} x^{\frac{5}{12}} - 4 x^{\frac{1}{4}} + 12 x^{\frac{1}{2}} - 12 \tan^{-1} x^{\frac{1}{12}} + C.$$

لا نفع في التوقف ، ولا نفع في المراجعة ، يكفينا ذلك

Graves

الحياة - بسيطة

$$(12) \int \frac{dt}{\sec^2 t + \tan^2 t}$$

$$u = \tan t \rightarrow du = \sec^2 t dt.$$

$$\int \frac{du}{\sec^2 t (\sec^2 t + \tan^2 t)} \rightarrow \int \frac{du}{(1+u^2)(1+2u^2)}.$$

$$\frac{1}{(1+u^2)(1+2u^2)} = \frac{Au+B}{1+u^2} + \frac{Cu+D}{1+2u^2}.$$

$$1 = (Au+B)(1+2u^2) + (Cu+D)(1+u^2)$$

$$1 = Au + 2Au^3 + B + 2Bu^2 + Cu + Cu^3 + D + Du^2.$$

$$1 = (2A+C)u^3 + (2B+D)u^2 + (A+C)u + (B+D)$$

$$B+D = 1 \rightarrow B = 1-D$$

$$A+C = 0 \rightarrow A = -C.$$

$$2B+D = 0, \quad 2A+C = 0.$$

$$A = 0, \quad C = 0, \quad B = -1, \quad D = 2.$$

$$\int \frac{du}{(1+u^2)(1+2u^2)} = -\int \frac{1}{1+u^2} du + 2 \int \frac{1}{1+2u^2} du.$$

$$I = -\tan^{-1} u + \frac{2}{\sqrt{2}} \tan^{-1} \sqrt{2}u + C.$$

* تكامل بطريقة أعمال المربع:

إذا احتوى التكامل على المقدار $ax^2 + bx + c$ ولأنه هذا المقدار غير قابل للتكامل، فإننا نقوم بعملية أعمال المربع، وذلك بجمع و طرح $(\frac{1}{2} \text{ معامل } x)^2$ ، ويشترط معامل $x^2 = 1$.

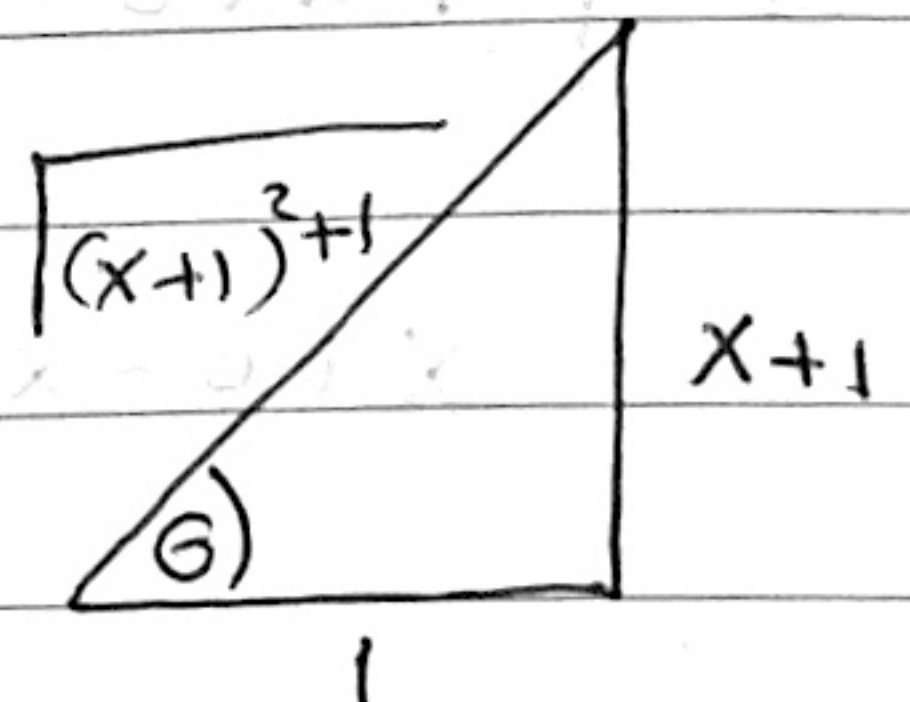
Q₁₅: Find the following int.

$$\textcircled{1} \int \frac{dx}{\sqrt{x^2 + 2x + 2}} = \int \frac{dx}{\sqrt{x^2 + 2x + 1 - 1 + 2}}$$

$$\int \frac{dx}{\sqrt{(x+1)^2 + 1}}$$

تقويضات مثلثية

$$x+1 = \tan \theta \rightarrow dx = \sec^2 \theta d\theta$$



$$I = \int \frac{\sec^2 \theta}{\sqrt{\tan^2 \theta + 1}} d\theta \rightarrow \int \frac{\sec^2 \theta}{\sec \theta} d\theta$$

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$

$$I = \ln |\sqrt{(x+1)^2 + 1} + x+1| + C$$

$$\textcircled{2} \int \frac{dx}{\sqrt{3+2x-x^2}} \rightarrow 3+2x-x^2 = -(x^2-2x-3)$$

$$= -(x^2-2x+1-1-3)$$

$$= -((x^2-1)^2 - 4) \rightarrow 4 - (x^2-1)^2$$

$$I = \int \frac{dx}{\sqrt{4 - (x-1)^2}}$$

$$x-1 = 2 \sin \theta \rightarrow dx = 2 \cos \theta d\theta.$$

$$I = \int \frac{2 \cos \theta d\theta}{\sqrt{4 - 4 \sin^2 \theta}} \rightarrow \int \frac{2 \cos \theta}{2 \cos \theta} d\theta.$$

$$\int d\theta = \theta + C = \sin^{-1}\left(\frac{x-1}{2}\right) + C.$$

$$\textcircled{3} \int \sqrt{x(6-x)} dx.$$

$$x(6-x) = 6x - x^2 = -(x^2 - 6x).$$

$$= -(x^2 - 6x + 9 - 9) = -((x^2 - 6x + 9) - 9),$$

$$= -((x-3)^2 - 9) = 9 - (x-3)^2.$$

$$\int \sqrt{9 - (x-3)^2} dx.$$

$$x-3 = 3 \sin \theta \rightarrow dx = 3 \cos \theta d\theta.$$

$$\int \sqrt{9 - 9 \sin^2 \theta} \cdot 3 \cos \theta d\theta.$$

$$\int 9 \cos^2 \theta d\theta \rightarrow 9 \int \frac{1}{2} (1 + \cos 2\theta) d\theta.$$

$$I = \frac{9}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + c.$$

$$= \frac{9}{2} (\theta + \sin \theta \cos \theta) + c.$$

$$I = \frac{9}{2} \left(\sin^{-1} \frac{x-3}{3} + \frac{x-3}{3} \cdot \frac{\sqrt{9-(x-3)^2}}{3} \right) + c.$$

$$\textcircled{4} \int \frac{dy}{(2y+1)\sqrt{y^2+y}}$$

$$y^2 + y = y^2 + y + \frac{1}{4} - \frac{1}{4} = \left(y + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

$$\int \frac{dy}{(2y+1)\sqrt{\left(y+\frac{1}{2}\right)^2 - \frac{1}{4}}} \quad \text{let } u = y + \frac{1}{2} \Rightarrow du = dy$$

$$\int \frac{du}{2u\sqrt{u^2 - \frac{1}{4}}}$$

$$y + \frac{1}{2} = \frac{1}{2} \sec \theta \rightarrow dy = \frac{1}{2} \sec \theta \tan \theta \cdot d\theta.$$

$$I = \int \frac{\frac{1}{2} \sec \theta \tan \theta \cdot d\theta}{2 \cdot \frac{1}{2} \sec \theta \sqrt{\frac{1}{4} \sec^2 \theta - \frac{1}{4}}} = \int \frac{\tan \theta \cdot d\theta}{2 \cdot \frac{1}{2} \tan \theta}$$

$$I = \int d\theta \rightarrow \theta + c. \rightarrow \sec^{-1} \frac{y+\frac{1}{2}}{\frac{1}{2}} + c.$$

$$\sec^{-1} 2y+1 + c.$$

$$\textcircled{5} \int \frac{x^3 + 2}{x^2 + 2x + 2} dx$$

$$\frac{x^3 + 2}{x^2 + 2x + 2} = x - 2 + \frac{2x+6}{x^2+2x+2}$$

$$\begin{array}{r} x^3 + 2 \\ -(x^2 + 2x + 2) \\ \hline -2x^2 - 2x + 2 \\ -(-2x^2 - 4x - 4) \\ \hline 2x + 6 \end{array}$$

⑥

$$\int \left(x - 2 + \frac{2x+6}{x^2+2x+2} \right) dx \rightarrow \frac{1}{2} x^2 - 2x + \int \frac{2x+6}{x^2+2x+2} dx.$$

$$x^2+2x+1-1+2 = (x+1)^2+1.$$

$$I = \frac{x^2}{2} - 2x + \int \frac{2x+6}{(x+1)^2+1} dx$$

$$x+1 = \tan \theta \rightarrow dx = \sec^2 \theta d\theta.$$

$$I = \frac{x^2}{2} - 2x + 2 \int \frac{\tan \theta - 1 + 3}{\tan^2 \theta + 1} \sec^2 \theta d\theta.$$

$$I = \frac{x^2}{2} - 2x + 2(\ln |\sec \theta| + 2\theta) + C.$$

$$I = \frac{x^2}{2} - 2x + 2(\ln |\sqrt{(x+1)^2+1}| + 2\tan^{-1}(x+1)) + C$$

$$\textcircled{6} \int \frac{dx}{x^2+4x+6} \rightarrow \frac{1}{(x+2)^2+2} \rightarrow \frac{1}{\left(\frac{x+2}{\sqrt{2}}\right)^2+1}$$

$$\int \frac{1}{(x+2)^2+2} dx = \frac{1}{2} \int \frac{1}{\left(\frac{x+2}{\sqrt{2}}\right)^2+1} dx$$

$$I = \frac{1}{\sqrt{2}} \tan^{-1} \frac{x+2}{\sqrt{2}} + C.$$

$$\textcircled{7} \int \frac{\sin x}{\tan x + \cos x} dx \rightarrow \int \frac{\sin x}{\frac{\sin x}{\cos x} + \cos x} dx$$

$$\int \frac{\sin x \cos x}{\sin x + \cos^2 x} dx$$

$$u = \sin x \rightarrow du = \cos x dx$$

$$\int \frac{u du}{u + 1 - u^2} \rightarrow \int \frac{u}{-u^2 + u + 1} du$$

$$-(u^2 - u - 1) = -(u^2 - u + \frac{1}{4} - \frac{1}{4} - 1)$$

$$= -\left[\left(u - \frac{1}{2}\right)^2 - \frac{5}{4}\right] = \frac{5}{4} - \left(u - \frac{1}{2}\right)^2$$

$$\int \frac{u}{\frac{5}{4} - \left(u - \frac{1}{2}\right)^2} du$$

$$u - \frac{1}{2} = \frac{\sqrt{5}}{2} \sin \theta \rightarrow u = \frac{\sqrt{5}}{2} \sin \theta + \frac{1}{2}$$

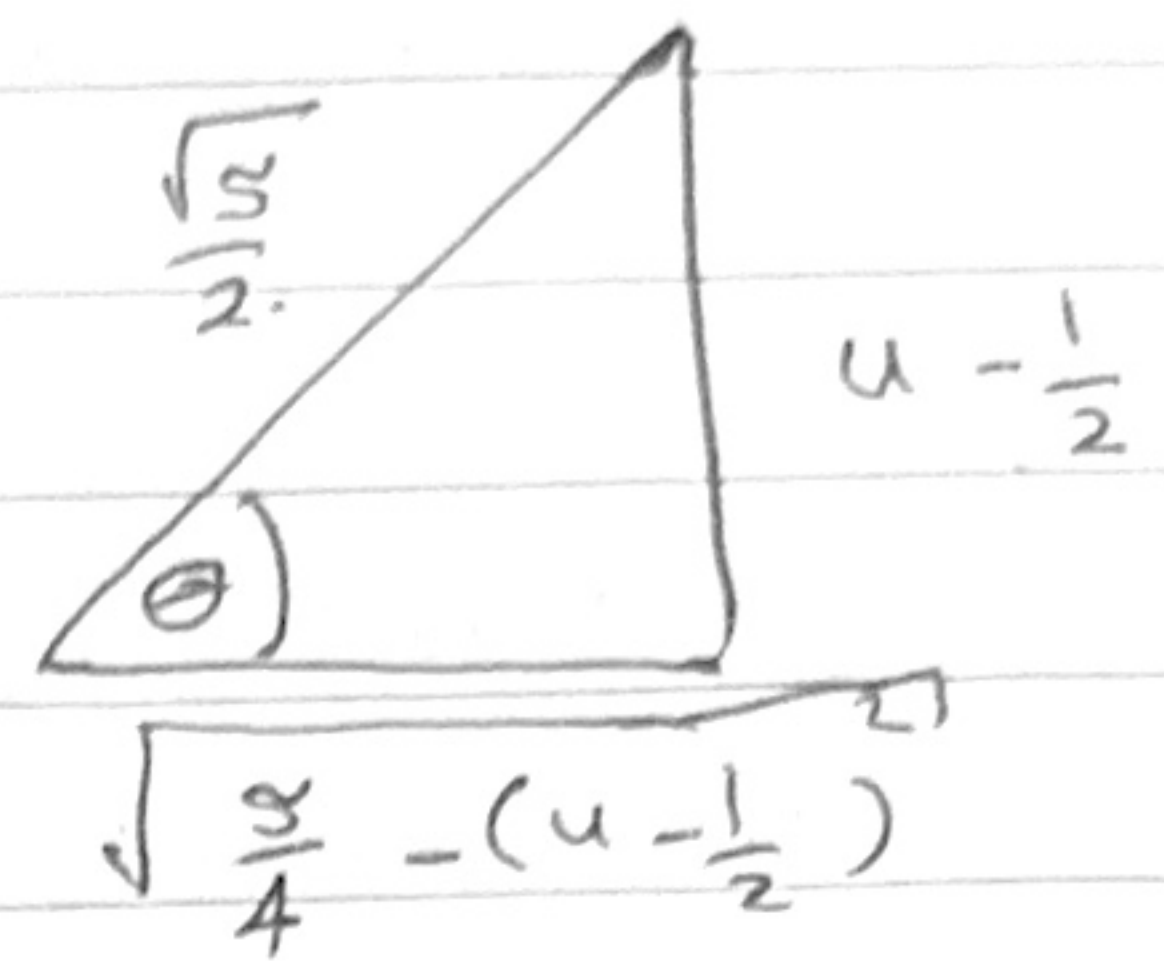
$$du = \frac{\sqrt{5}}{2} \cos \theta d\theta$$

$$\int \frac{\frac{\sqrt{5}}{2} \sin \theta + \frac{1}{2} \cdot \frac{\sqrt{5}}{2} \cos \theta d\theta}{\frac{5}{4} - \frac{5}{4} \sin^2 \theta}$$

$$\int \frac{\frac{\sqrt{5}}{2} \sin \theta + \frac{1}{2} (\frac{\sqrt{5}}{2}) d\theta}{\cos \theta}$$

⑥

$$\frac{2}{\sqrt{s}} \int \frac{\frac{\sqrt{s}}{2} \sin \theta + \frac{1}{2}}{\cos \theta} d\theta$$



$$I = \frac{2}{\sqrt{s}} \int \left(\frac{\sqrt{s}}{2} \tan \theta + \frac{1}{2} \sec \theta \right) d\theta.$$

$$I = \frac{2}{\sqrt{s}} \left(\frac{\sqrt{s}}{2} \ln |\sec \theta| + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) + C$$

$$I = \frac{2}{\sqrt{s}} \left(\frac{\sqrt{s}}{2} \ln \left| \frac{\sqrt{s}/2}{\sqrt{s/4 - (u - 1/2)^2}} \right| + \frac{1}{2} \ln \left| \frac{s/2}{\sqrt{s/4 - (u - 1/2)^2}} + \frac{u - 1/2}{\sqrt{s/4 - (u - 1/2)^2}} \right| \right) + C$$

$$I = \frac{2}{\sqrt{s}} \left(\frac{\sqrt{s}}{2} \ln \left| \frac{\sqrt{s}}{2 \sqrt{s/4 - (\sin x)}} \right| \right) \rightarrow \text{أعمل الباقي}$$

التعويض بدلالة من (u) ما مضى (x)

* تكامل الدوال الفبائية المحتوي على دوال مثلثية ($\sin x, \cos x$)

إن التعويض $z = \tan\left(\frac{x}{2}\right)$ يمكننا من تحويل أي دالة مسألة

تتكامل فبائية في $\sin x, \cos x$ إلى مسألة تحتوي على دالة فبائية

في (z) كالتالي:

① نفرض في المسألة التكامل بالتعويضات الآتية:

$$\sin x = \frac{2z}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$dx = \frac{2dz}{1+z^2}$$

② نكتب مسألة التكامل التي تكامل الدالة باستخدام قوانين أو طرق التكامل.

③ نفرض ب $z = \tan\left(\frac{x}{2}\right)$ بعد إيجاد تكامل الدالة.

Q₁₆: Find the following int.

$$\textcircled{1} \int \frac{1}{1+\sin x} dx \rightarrow \int \frac{1}{1+\frac{2z}{1+z^2}} \cdot \frac{2dz}{1+z^2}$$

$$\int \frac{1}{\frac{z^2+2z+1}{1+z^2}} \cdot \frac{2dz}{1+z^2}$$

$$\int \frac{2dz}{z^2+2z+1} \rightarrow 2 \int \frac{1}{(z+1)^2} dz.$$

$$2 \int (z+1)^{-2} dz = -\frac{2}{z+1} + C.$$

$$I = -\frac{2}{\tan(\frac{x}{2})+1} + C.$$

$$\textcircled{2} \int \frac{1}{1+\cos x} dx \rightarrow \int \frac{1}{1+\frac{1-z^2}{1+z^2}} \cdot \frac{2dz}{1+z^2}.$$

$$\int \frac{1}{\frac{z^2+1+1-z^2}{1+z^2}} \cdot \frac{2dz}{1+z^2}.$$

$$\int \frac{2}{2} dz \rightarrow \int dz = z + C.$$

$$I = \tan\left(\frac{x}{2}\right) + C.$$

تفاريق متنوعة

$$① \int \frac{dx}{x^3 + x^2 + x + 1}$$

$$x^3 + x^2 + x + 1 = x(x^2 + 1) + (x^2 + 1) = (x^2 + 1)(x + 1).$$

$$\int \frac{dx}{(x^2 + 1)(x + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1}$$

$$1 = (Ax + B)(x + 1) + C(x^2 + 1)$$

$$1 = Ax^2 + Ax + Bx + B + Cx^2 + C.$$

$$A + C = 0, \quad A + B = 0, \quad B + C = 1.$$

$$A = -\frac{1}{2}, \quad B = \frac{1}{2}, \quad C = \frac{1}{2}.$$

$$I = \int \frac{dx}{(x^2 + 1)(x + 1)} = -\frac{1}{2} \int \frac{x - 1}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x + 1} dx.$$

$$I = -\frac{1}{4} \int \frac{2x}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x + 1} dx.$$

$$I = -\frac{1}{4} \ln |x^2 + 1| + \frac{1}{2} \tan^{-1}(x) + \frac{1}{2} \ln |x + 1| + C.$$

$$\textcircled{2} \int \frac{\sec^2 x (\sec^2 x + 1)}{\tan^3 x + 1} dx.$$

$$\int \frac{\sec^2 x (\tan^2 x + 2)}{\tan^3 x + 1} dx. \quad u = \tan x, \quad du = \sec^2 x dx.$$

$$\int \frac{u^2 + 2}{u^3 + 1} du \rightarrow \int \frac{u^2 + 2}{(u+1)(u^2 - u + 1)} du.$$

$$u^2 + 2 = A(u^2 - u + 1) + (Bu + C)(u+1)$$

$$\text{at } u = -1$$

$$\boxed{A = 3}$$

$$u^2 + 2 = Au^2 - Au + A + Bu^2 + Bu + Cu + C.$$

$$A + C = 2. \rightarrow \boxed{C = 1}$$

$$-A + B + C = 0. \rightarrow \boxed{B = 0}$$

$$\int \frac{u^2 + 2}{(u+1)(u^2 - u + 1)} du = 3 \int \frac{1}{u+1} du + \int \frac{1}{u^2 - u + 1} du.$$

آلعمال المخرج

$$I = \int \frac{1}{u^2 - u + 1} du \rightarrow \int \frac{du}{(u - \frac{1}{2})^2 + \frac{3}{4}}$$

$$I = \frac{2}{\sqrt{3}} \int \frac{\frac{2}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}\left(u - \frac{1}{2}\right)\right)^2 + 1} du = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(u - \frac{1}{2}\right)\right) + C.$$

$$I = \int \frac{u^2 + 2}{(u+1)(u^2 - u + 1)} du = \ln|u+1| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(u - \frac{1}{2}\right)\right) + C.$$

$$I = \int \frac{\sec^2 x (\sec^2 x + 1)}{\tan^3 x + 1} dx = \ln|\tan x + 1| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\tan x - \frac{1}{2}\right)\right) + C.$$

$$\textcircled{3} \int \frac{dx}{\sin x + \tan x} \rightarrow \int \frac{\cos x}{\sin x \cos x + \sin x} dx.$$

$$\int \left[\frac{\frac{1-z^2}{1+z^2}}{\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2}} \right] + \frac{2dz}{1+z^2}.$$

$$\int \frac{\frac{1-z^2}{1+z^2}}{\frac{z-z^3}{1+z^2} - z} dz \rightarrow \int \frac{\frac{1-z^2}{1+z^2}}{\frac{2z}{1+z^2}} dz.$$

$$\int \frac{1-z^2}{2z} dz \rightarrow \frac{1}{2} \int \left(\frac{1}{z} - z \right) dz.$$

$$I = \left[\frac{1}{2} \ln|z| - \frac{1}{4} z^2 \right] + C \rightarrow \frac{1}{2} \ln\left|\tan\left(\frac{x}{2}\right)\right| - \frac{1}{4} \tan^2 \frac{x}{2} + C.$$

$$\textcircled{4} \int \frac{x}{\sqrt{x+1}-1} dx.$$

$$u = (x+1)^{\frac{1}{2}} \rightarrow x+1 = u^2 \rightarrow dx = 2u du.$$

$$\int \frac{(u^2-1) \cdot 2u du}{u-1} \rightarrow 2 \int \frac{u(u+1)(u-1) du}{(u-1)}.$$

$$2 \int (u^2 + u) du \rightarrow 2 \left[\frac{u^3}{3} + \frac{u^2}{2} \right] + C$$

$$\frac{2}{3} u^3 + u^2 + C = \frac{2}{3} \sqrt{(x+1)^3} + x+1 + C.$$

$$\textcircled{5} \int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

$$z = x^{\frac{1}{6}}, \quad x = z^6 \rightarrow dx = 6z^5 dz.$$

$$\int \frac{6z^5 dz}{z^3 + z^2} \rightarrow \int \frac{6z^3}{z+1} dz.$$

$$\int \left(6z^2 - 6z + 6 - \frac{6}{z+1} \right) dz.$$

$$I = 2z^3 - 3z^2 + 6z - 6 \ln|z+1| + C.$$

$$I = \int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln|\sqrt[6]{x} + 1| + C.$$

$$\textcircled{6} \int \frac{\sqrt{x}}{\sqrt[4]{x} + x} dx.$$

$$z = x^{\frac{1}{4}}, \quad x = z^4 \rightarrow dx = 4z^3 dz.$$

$$\int \frac{z^2 \cdot 4z^3 dz}{z^4 + z} \rightarrow 4 \int \frac{z^4}{z^3 + 1} dz$$

$$I = 4 \int \left(z - \frac{z}{z^3 + 1} \right) dz, \rightarrow 4 \int z dz - 4 \int \frac{z}{z^3 + 1} dz$$

تجزیہ جزئیہ

$$* I = \int \frac{z}{z^3 + 1} dz = \int \frac{z}{(z+1)(z^2 - z + 1)} dz.$$

$$z = \frac{A}{z+1} + \frac{Bz + C}{z^2 - z + 1}$$

$$\text{at } z = -1$$

$$\boxed{A = -\frac{1}{3}}$$

$$-A + B + C = -1$$

$$A + B = 0 \rightarrow$$

$$\boxed{B = \frac{1}{3}}$$

$$\boxed{C = \frac{1}{3}}$$

$$I = \int \frac{dz}{(z+1)(z^2-z+1)} = -\frac{1}{3} \int \frac{dz}{z+1} + \frac{1}{3} \int \frac{z-2}{z^2-z+1} dz$$

$$I_1 = \int \frac{z-2}{z^2-z+1} dz = \frac{1}{2} \int \frac{2z-1-3}{z^2-z+1} dz$$

آمال مربع

$$\frac{1}{2} \int \frac{2z-1}{z^2-z+1} dz - \frac{3}{2} \int \frac{1}{z^2-z+1} dz$$

آمال مربع آخر

$$I_2 = \int \frac{dz}{z^2-z+1} = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(z - \frac{1}{2} \right) \right) \rightarrow \text{الطريقة موجودة في 68}$$

$$I_1 = \frac{1}{2} \int \frac{2z-1}{z^2-z+1} dz - \frac{3}{2} I_2$$

$$I_1 = \frac{1}{2} \ln |z^2-z+1| - \sqrt{3} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(z - \frac{1}{2} \right) \right)$$

$$I = -\frac{1}{3} \int \frac{dz}{z+1} + \frac{1}{3} I_1$$

$$I = -\frac{1}{3} \ln |z+1| + \frac{1}{6} \ln |z^2-z+1| - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(z - \frac{1}{2} \right) \right)$$

$$I = \int \frac{\sqrt{x}}{4\sqrt{x}+x} dx = 4 \int \frac{z}{z^2+1} dz - 4I$$

$$= 2\sqrt{x} + \frac{4}{3} \ln |4\sqrt{x}+1| - \frac{2}{3} \ln |\sqrt{x}-4\sqrt{x}+1| + \frac{4}{3} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(\sqrt{x} - \frac{1}{2} \right) \right) + C$$