التكامل

مفهوم التكامل الغير محدد :

درسنا في التفاضل عندما تكون لدينا دالة ما f فإننا نستطيع أن نحصل على دالة أخرى تسمى المشتقة الأولى للدالة f ورمزنا لها بالرمز $\acute{f}$. غير أنه في كثير من المواقف العملية يحدث العكس ، أي تكون لدينا $\acute{f}$ ونرغب في معرفة الدالة الأصلية f ، كأن يكون لدينا ميل المماس عند أي نقطة على منحنى ما ونود معرفة معادلة هذا المنحنى ، لهذا نحتاج إلى البحث عن دالة إذا فاضلناها حصلنا على المشتقة المعلومة .

تعريف :

إذا كانت $f(x)$ دالة متصلة وأمكن إيجاد دالة $F(x)$ قابلة للإشتقاق عند كل نقطة في نطاقها بحيث كان $\acute{F}(x) = f(x)$ فإن $F(x)$ تسمي المشتقة العكسية للدالة $f(x)$

$$\boxed{F(x) = \int f(x)\,dx \overset{\text{إذا وفقط إذا كان}}{\Longleftrightarrow} \acute{F}(x) = f(x)}$$

$$\frac{d(x^3)}{dx} = 3x^2 \implies \int 3x^2\,dx = x^3 + c$$

$$\frac{d(3x^2-5)}{dx} = 6x \implies \int 6x\,dx = 3x^2 + c$$

$$\frac{d(\cos 2x)}{dx} = \frac{1}{2}\sin 2x \implies \int \frac{1}{2}\sin 2x\,dx = \cos 2x + c$$

$$\frac{d(e^{2x})}{dx} = 2e^{2x} \implies \int 2e^{2x}\,dx = e^{2x} + c$$

بصورة عامة :

$$\frac{d[F(x)]}{dx} = f(x) \implies \int f(x)\,dx = F(x) + c$$

الصيغ الاساسية للتكامل :

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad , \quad n \neq -1$

2. $\int k\,dx = k\,x + C$

3. $\int \frac{1}{x}dx = \ln|x| + C$

أمثلة :

1) $\int x^5\,dx = \frac{x^6}{6} + C$

2) $\int 5\ dx = 5x + C$

3) $\int \frac{3}{x}dx = 3\int \frac{1}{x}dx = 3\ln|x| + C$

4) $\int \frac{1}{x\sqrt{x}}dx = \int \frac{1}{x^{\frac{3}{2}}}dx = \int x^{-\frac{3}{2}}dx = -2\,x^{-\frac{1}{2}} + C = -\frac{2}{\sqrt{x}} + C$

نظرية :

1. $\frac{d}{dx}\int f(x)\,dx = f(x)$

2. $\int \frac{d}{dx}[f(x)]\,dx = f(x) + C$

3. $\int k\ \ f(x)\,dx = k\ \ f(x) + C$

4. $\int [f(x) + g(x)]\,dx = \int f(x)\ dx \pm \int g(x)\ dx$

5) $\frac{d}{dx}\int x^5\,dx = x^5$

6) $\frac{d}{dx}\int (x + \sqrt{\tan x}\ )^5\,dx = (x + \sqrt{\tan x}\ )^5$

7) $\int \frac{d}{dx}\left(\sqrt{x^5 + 3}\right)dx = \sqrt{x^5 + 3} + C$

8) $\int (6\,x^2 - 2x + 1)\ dx = 6\int x^2\,dx - 2\int x\,dx + \int 1\,dx$

$$= 6 \times \frac{x^3}{3} - 2 \times \frac{x^2}{2} + x + C = 2x^3 - x^2 + x + C$$

9) $\int (x+2)(3x-2)dx = \int (3x^2+4x-4)dx = x^3+2x^2-4x+C$

10) $\int \left(2-\sqrt{x}\right)^2 dx = \int \left(4-4\sqrt{x}+x\right)dx = 4x - 4\frac{2\,x^{\frac{3}{2}}}{3} - \frac{x^2}{2} + C$

$$= 4x - \frac{8\,x\sqrt{x}}{3} - \frac{x^2}{2} + C$$

11) $\int \frac{3}{x}dx = 3\int \frac{1}{x}dx = 3\ln x + C$

12) $\int \frac{(x+1)^3}{x}dx = \int x^{-1}(x^3+3x^2+3x+1)\ dx = \int \left(x^2+3x+3+\frac{1}{x}\right)dx$

مقام واحد نقسم على x الحل

$$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 3x + \ln x + C$$

13) $\int \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2 dt = \int \left(t-2+\frac{1}{t}\right)dt = \frac{1}{2}t^2 - 2t + \ln t + C$

نفرد سادي 1 بعد فكه

14) $\int \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)dt = \int \left(t-\frac{1}{t}\right)dt = \frac{1}{2}t^2 - \ln t + C$

15) $\int \frac{\sqrt{v}-v\sqrt{v}}{v^2}dv = \int v^{-2}\left(v^{\frac{1}{2}}-v^{\frac{3}{2}}\right)dv = \int \left(v^{-\frac{3}{2}}-v^{-\frac{1}{2}}\right)dv$

$$= -2v^{-\frac{1}{2}} - 2v^{\frac{1}{2}} + c = -2\left(\frac{1}{\sqrt{v}}+\sqrt{v}\right) + c$$

$\frac{1}{2} - \frac{2}{2}$ نطرح الاس

$\frac{3}{2} - \frac{2}{2}$ نطرح الاس

ملاحظة :-

$(a+b)^3$

$= a^3 + 3a^2b + 3b^2a + \dots$

$(a-b)^3$

$= a^3 - 3a^2b + 3b^2a - \dots$

مثال ن 13)

$\int \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2 dx$

معناها قوس ثابت

الحل $\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2 x + C$

التكامل المحدد :

العلاقة بين التكامل المحدد والتكامل الغير محدد

إذا كانت $f(x)$ دالة متصلة على الفترة $[a \text{ ، } b]$ وكانت $f(x) = \acute{G}(x)$ فإن :

$$\int_a^b f(x)\,dx = [G(x)]_a^b = G(b) - G(a)$$

أمثلة :

1) $\displaystyle\int_{-1}^{2} (2x-3)\,dx = [x^2 - 3x]_{-1}^{2} = (4-6) - (1+3) = -6$

2) $\displaystyle\int_{1}^{4} \left(x\sqrt{x} + \frac{3}{x}\right) dx = \int_{1}^{4} \left(x^{\frac{3}{2}} + \frac{3}{x}\right) dx = \left[\frac{2}{5}x^{\frac{5}{2}} + 3\ln|x|\right]_1^4$

$$= \left(\frac{2}{5} \times 4^{\frac{5}{2}} + 3\ln 4\right) - \left(\frac{2}{5} + 3\ln 1\right) = \frac{64}{5} + 3\ln 4 - \frac{2}{5} = \frac{62}{5} + 3\ln 4$$

3) $\displaystyle\int_{0}^{4} (4-t)\sqrt{t}\;dt = \int_{0}^{4} \left(4\,t^{\frac{1}{2}} - t^{\frac{3}{2}}\right) dx = \left[\frac{8}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^4$

$$= \left(\frac{64}{3} - \frac{64}{5}\right) - (0-0) = \frac{128}{15}$$

3) $\displaystyle\int_{1}^{2} \frac{v^3 + 3v^6}{v^4}\,dv = \int_{1}^{2} \left(\frac{1}{v} + 3v^2\right)\,dv = [\ln|v| + v^3]_1^2$

$$= (\ln 2 + 8) - (\ln 1 - 1) = \ln 2 - 8 + 1 = \ln 2 + 7$$

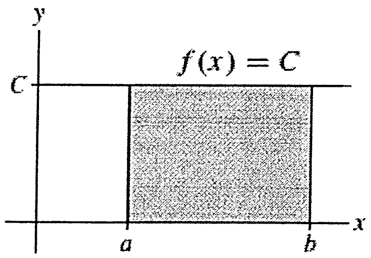
Ln 1=0

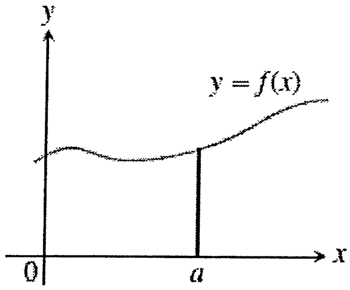
إعداد/ أ.يحيي طه

طرق التكامل

خواص التكامل المحدد :

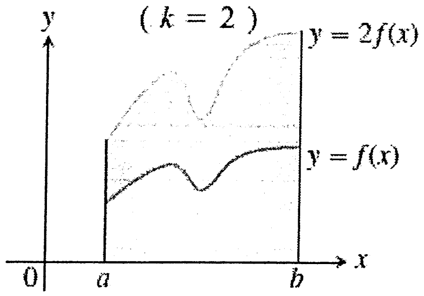
$$1 \bullet \int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

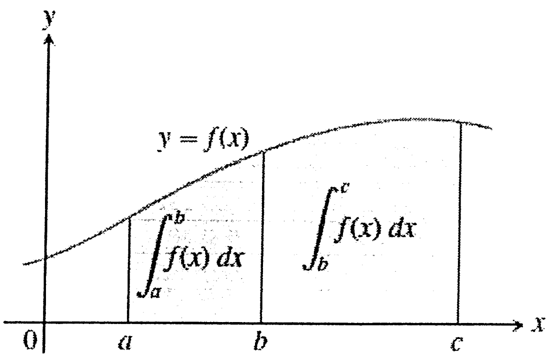




$$2 \bullet \int_a^b c\ dx = c(b-a)$$

$$3 \bullet \int_a^a f(x)\ dx = 0 \quad (0 = \text{المساحة})$$

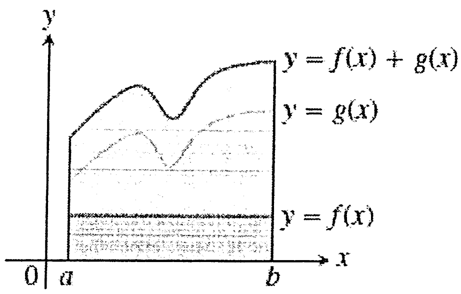




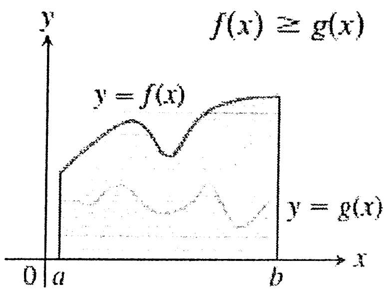
$$4 \bullet \int_a^b c\, f(x)\ dx = c\int_a^b f(x)\ dx$$

$$5 \bullet \int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

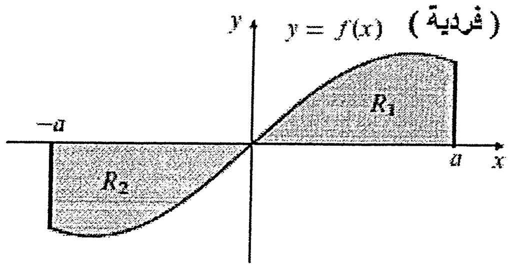
$$6 \bullet \int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$



$$7 \bullet f(x) \geq g(x) \text{ on } [a ، b] \Rightarrow \int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$$

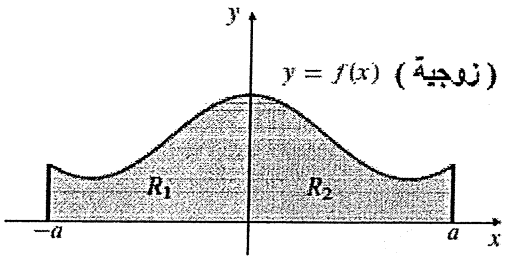


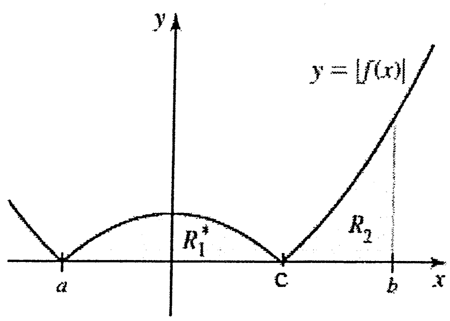
$$8 \bullet \int_{-a}^{a} f(x)\ dx = 0$$



$$9 \bullet \int_{-a}^{a} f(x)\ dx = \int_{-a}^{0} f(x)\ dx + \int_{0}^{a} f(x)\ dx$$

$$= 2\int_{-a}^{0} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$





$$10 \bullet \int_{a}^{b} |f(x)|\ dx = \int_{a}^{c} f(x)\ dx + \int_{c}^{b} f(x)\ dx$$

$$11 \bullet \frac{d}{dx}\int_{a}^{b} f(x)\, dx = 0$$

تفاضل وتكامل محدد

$$12 \bullet \int_{a}^{b} \frac{d}{dx} f(x)\, dx = f(b) - f(a)$$

$\int_{0}^{2} \frac{d}{dx}\sqrt{x^3+1}\ dx \to \sqrt{8+1} - \sqrt{0+1}$

أمثلة على هذه الخواص /

$$1)\ \int_{-1}^{2} (2x-3)\, dx = -6 \Longrightarrow \int_{2}^{-1} (2x-3)\, dx = 6$$

$$2)\ \int_{-1}^{4} 5\ dx = 5\big(4-(-1)\big) = 25$$

$$3)\ \int_{4}^{4} (2x-3)\ dx = 0$$

$$4)\ \int_{-1}^{2} 5(2x-3)\, dx = 5\int_{-1}^{2} (2x-3)\, dx = 5(-6) = -30$$

5) $\displaystyle\int_{-1}^{2}(2x-3)\,dx = \int_{-1}^{2}(2x)\,dx - \int_{-1}^{2}(3)\,dx$

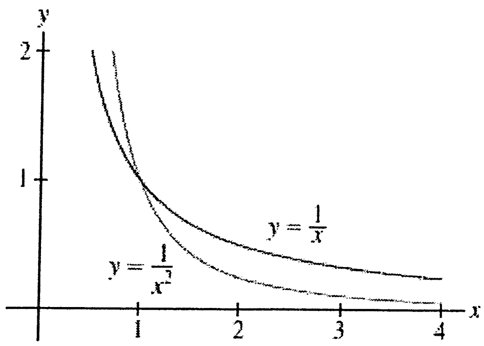
$$= [x^2]_{-1}^{2} - [3x]_{-1}^{2} = (4-1) - \big(6-(-3)\big) = 3 - 9 = -6$$

6) $\displaystyle\int_{-1}^{2}(2x-3)\,dx = \int_{-1}^{0}(2x-3)\,dx + \int_{0}^{2}(2x-3)\,dx$

$$= [x^2-3x]_{-1}^{0} + [x^2-3x]_{0}^{2}$$

$$= [(0-0)-(1+3)] + \big(4-6) - (0-0)\big) = -4 - 2 = -6$$

7) $x^2 > x \Longrightarrow \dfrac{1}{x} > \dfrac{1}{x^2}$ $on\ [1 \text{ ، } 4]$



$$\int_{1}^{4}\frac{1}{x}\,dx = [\ln|x|]_{1}^{4} = \ln 4 - \ln 1 = \ln 4 - 0 = \ln 4$$

$$\int_{1}^{4}\frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_{1}^{4} = \left(\frac{-1}{4}\right) - \left(\frac{-1}{1}\right) = \frac{3}{4}$$

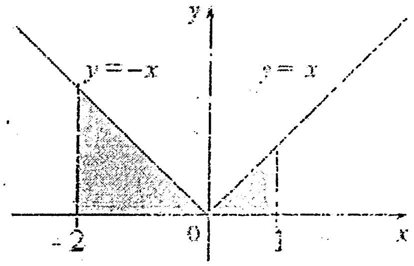
$$\therefore \int_{1}^{4}\frac{1}{x}\,dx > \int_{1}^{4}\frac{1}{x^2}$$

8) $f(x) = x^3$ دالة فردية $\Longrightarrow \displaystyle\int_{-2}^{2}x^3\,dx = \left[\frac{x^4}{4}\right]_{-2}^{2} = \frac{16}{4} - \frac{16}{4} = 0$

9) $f(x) = x^2$ دالة زوجية $\Longrightarrow \displaystyle\int_{-2}^{2}x^2\,dx = \left[\frac{x^3}{3}\right]_{-2}^{2} = \frac{8}{3} - \frac{-8}{3} = \frac{16}{3}$

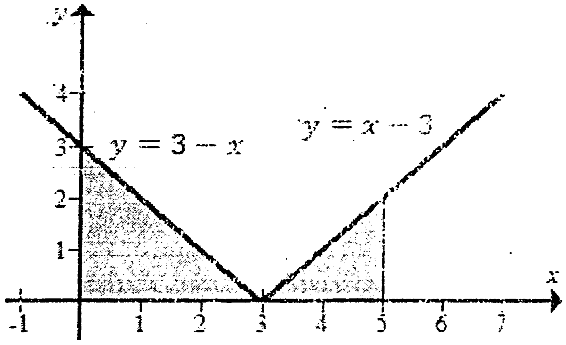
$$\Longrightarrow \int_{-2}^{2}x^2\,dx = 2\int_{0}^{2}x^2\,dx = 2\left[\frac{x^3}{3}\right]_{0}^{2} = 2\left[\frac{8}{3} - 0\right] = \frac{16}{3}$$

$$10)\quad \int_{-2}^{1} |x|\,dx = \int_{-2}^{0} -x\,dx + \int_{0}^{1} x\,dx$$



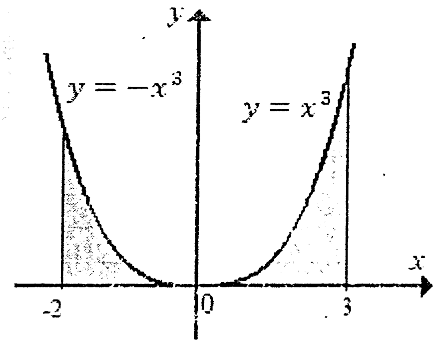
$$= \left[\frac{-x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1}$$

$$= \left[0 - \frac{-4}{2}\right] + \left[\frac{1}{2} - 0\right] = 2 + \frac{1}{2} = \frac{5}{2}$$



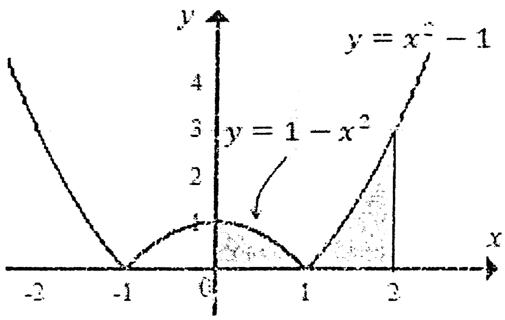
$$11)\quad \int_{0}^{5} |3 - x|\,dx = \int_{0}^{3} (3 - x)\,dx + \int_{3}^{5} (x - 3)\,dx$$

$$= \left[3x - \frac{1}{2}\,x^2\right]_{0}^{3} + \left[\frac{1}{2}\,x^2 - 3x\right]_{3}^{5}$$



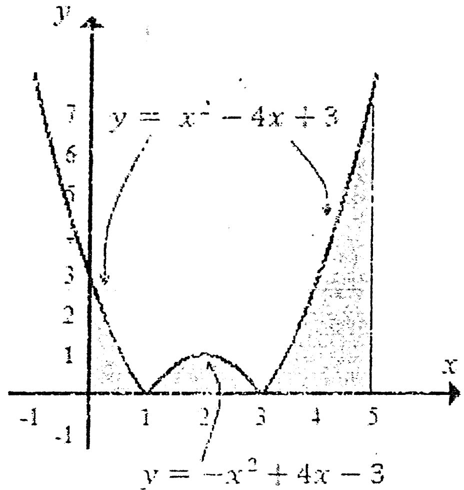
$$= \left(9 - \frac{9}{2}\right) - (0) + \left(\frac{25}{2} - 15\right) - \left(\frac{9}{2} - 9\right) = \frac{13}{2}$$

$$12)\quad \int_{-2}^{3} |x^3|\,dx = \int_{-2}^{0} -x^3\,dx + \int_{0}^{3} x^3\,dx = 4 + \frac{81}{4} = \frac{97}{4}$$



$$13)\quad \int_{0}^{2} |x^2 - 1|\,dx = \int_{0}^{1} (x^2 - 1)\,dx + \int_{1}^{2} (1 - x^2)\,dx$$

$$= \frac{2}{3} + \frac{4}{3} = 2$$



$$14)\quad \int_{0}^{5} |x^2 - 4x + 3|\,dx = \int_{0}^{1} (x^2 - 4x + 3)\,dx$$

$$+ \int_{1}^{3} (-x^2 + 4x - 3)\,dx + \int_{3}^{5} (x^2 - 4x + 3)\,dx$$

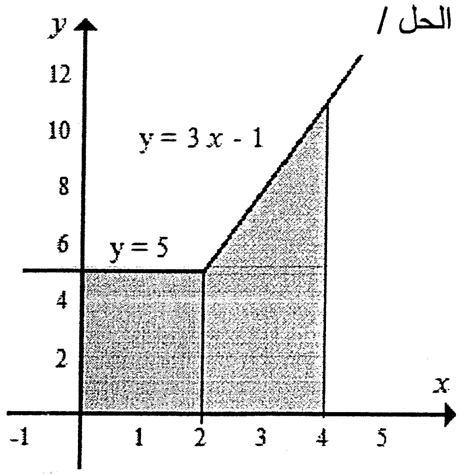
$$= \frac{4}{3} + \frac{4}{3} + \frac{20}{3} = \frac{28}{3}$$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3)=0$

15) أوجد $\int_{0}^{4} f(x)\,dx$ عندما $f(x)=\begin{cases}5 & if\ x\le 2\\ 3x-1 & if\ x\ge 2\end{cases}$

الحل /



$$\int_{0}^{4} f(x)\,dx=\int_{0}^{2} f(x)\,dx+\int_{2}^{4} f(x)\,dx$$

$$=\int_{0}^{2} 5\ dx+\int_{2}^{4}(3x-1)\,dx=10+16=26$$

16) إذا كان $\int_{1}^{5} f(x)\,dx=-2$ ، $\int_{1}^{8} f(x)\,dx=10$ ، $\int_{1}^{5} g(x)\,dx=4$ أوجد :

$a)\ \int_{3}^{3} f(x)\ dx$ $\quad b)\ \int_{5}^{1} f(x)\ dx$ $\quad c)\ \int_{1}^{5} -2\,f(x)\,dx$

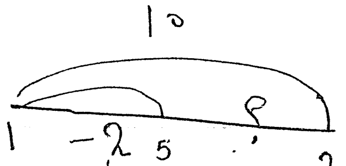
$d)\ \int_{5}^{8} f(x)\ dx$ $\quad e)\ \int_{1}^{5}[2\,f(x)+3g(x)]\,dx$

الحل /

$a)\ \int_{3}^{3} f(x)\ dx=0$ $\quad b)\ \int_{5}^{1} f(x)\ dx=-\int_{1}^{5} f(x)\,dx=-(-2)=2$

$$c)\ \int_{1}^{5} -2\,f(x)\,dx=-2\int_{1}^{5} f(x)\,dx=(-2)(-2)=4$$

$$d)\ \int_{5}^{8} f(x)\ dx=\int_{1}^{8} f(x)\ dx-\int_{1}^{5} f(x)\ dx=10-(-2)=12$$



10 -(-2) = 12

$$e)\ \int_{1}^{5}[2\,f(x)+3g(x)]\,dx=2\int_{1}^{5} f(x)\,dx+3\int_{1}^{5} g(x)\,dx=2(-2)+3(4)=8$$

نظرية الحدين الأعلى والأدنى لقيمة التكامل

إذا كان $m \le f(x) \le M$ ، $x \in [a \text{ ، } b]$ وكانت $f(x)$ قابلة للتكامل على الفترة $[a \text{ ، } b]$

فإن : $$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

مثال 1 / أوجد الحدين الأدنى والأعلى للتكامل $\int_1^3 \frac{2}{x+1}\,dx$

الحل /

$$\because\ 1 \le x \le 3 \Rightarrow 2 \le x+1 \le 4 \Rightarrow \frac{1}{2} \ge \frac{1}{x+1} \ge \frac{1}{4} \xRightarrow{\times 2} 1 \ge \frac{2}{x+1} \ge \frac{1}{2}$$

$$\because m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\Rightarrow \frac{1}{2}(3-1) \le \int_1^3 \frac{2}{x+1}\,dx \le 1(3-1) \Rightarrow 1 \le \int_1^3 \frac{2}{x+1}\,dx \le 2$$

$\therefore$ الحد الأدنى 1 والحد الأعلى 2

مثال 2 / أوجد الحدين الأدنى والأعلى للتكامل $\int_0^1 e^{-x^2}\,dx$

الحل /

$$\because\ 0 \le x \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow e^0 \le e^{x^2} \le e^1 \Rightarrow e^{-1} \le e^{-x^2} \le 1$$

$$\because m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\Rightarrow e^{-1}(1-0) \le \int_0^1 e^{-x^2}\,dx \le 1(1-0) \Rightarrow 0.368 \le \int_0^1 e^{-x^2}\,dx \le 1$$

مثال 3 / أثبت أن $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x \; dx \le \frac{\sqrt{3}\,\pi}{12}$

الحل /

$$\frac{\pi}{4} \le x \le \frac{\pi}{3} \Rightarrow \tan\frac{\pi}{4} \le \tan x \le \tan\frac{\pi}{3} \Rightarrow 1 \le \tan x \le \sqrt{3}$$

$$\Rightarrow 1\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x \; dx \le \sqrt{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \Rightarrow \frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x \; dx \le \frac{\pi\sqrt{3}}{12}$$

مثال 4 / أثبت أن $-\frac{\pi}{2} \le \int_{0}^{\pi/2} (\cos x - \sin x)\, dx \le \frac{\pi}{2}$

الحل /

$$\cos x - \sin x = \cos x - \cos\left(x - \tfrac{\pi}{2}\right) = -2\sin\left(\frac{x-(x-\frac{\pi}{2})}{2}\right)\sin\left(\frac{x+(x-\frac{\pi}{2})}{2}\right)$$

$$= -2\sin\left(\tfrac{\pi}{4}\right)\sin\left(x-\tfrac{\pi}{4}\right) = -\sqrt{2}\sin\left(x-\tfrac{\pi}{4}\right)$$

$$\because\; 0 \le x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{\pi}{4} \Rightarrow \sin(-\tfrac{\pi}{4}) \le \sin\left(x-\tfrac{\pi}{4}\right) \le \sin\tfrac{\pi}{4}$$

$$-\frac{\sqrt{2}}{2} \le \sin\left(x-\tfrac{\pi}{4}\right) \le \frac{\sqrt{2}}{2} \Rightarrow -1 \le \sqrt{2}\sin\left(x-\tfrac{\pi}{4}\right) \le 1$$

$$\Rightarrow 1 \ge -\sqrt{2}\sin\left(x-\tfrac{\pi}{4}\right) \ge -1 \Rightarrow -1\left(\frac{\pi}{2}-0\right) \le \int_{0}^{\pi/2} -\sqrt{2}\sin\left(x-\tfrac{\pi}{4}\right) dx \le 1\left(\frac{\pi}{2}-0\right)$$

$$\Rightarrow -\frac{\pi}{2} \le \int_{0}^{\pi/2} (\cos x - \sin x)\, dx \le \frac{\pi}{2}$$

تكامل الدوال المثلثية :

تحفظ

$1-\int \sin u \, du = -\cos u + c$

$2-\int \cos u \, du = \sin u + c$

$3-\int \sec^2 u \, du = \tan u + c$

$4-\int \csc^2 u \, du = -\cot u + c$

$5-\int \sec u \tan u \, du = \sec u + c$

$6-\int \csc u \cot u \, du = -\csc u + c$

$7-\int \tan u \, du = \ln|\sec u| + c$

$8-\int \cot u \, du = \ln|\sin u| + c$

$9-\int \sec u \, du = \ln|\sec u + \tan u| + c$

$10-\int \csc u \, du = \ln|\csc u - \cot u| + c$

* عندما تكون الزاوية خطية (من الدرجة الأولى)

$1-\int \sin(ax+b) = -\frac{1}{a}\cos(ax+b) + c$

$2-\int \cos(ax+b) \, du = \frac{1}{a}\sin(ax+b) + c$

$3-\int \sec^2(ax+b) \, du = \frac{1}{a}\tan(ax+b) + c$

$4-\int \csc^2(ax+b) \, du = -\frac{1}{a}\cot(ax+b) + c$

$5-\int \sec(ax+b)\tan(ax+b) \, du = \frac{1}{a}\sec(ax+b) + c$

$6-\int \csc(ax+b)\cot(ax+b) \, du = -\frac{1}{a}\csc(ax+b) + c$

$7-\int \tan(ax+b) \, du = \frac{1}{a}\ln|\sec(ax+b)| + c$

$8-\int \cot(ax+b) \, du = \frac{1}{a}\ln|\sin(ax+b)| + c$

$9-\int \sec(ax+b) \, du = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + c$

$10-\int \csc(ax+b) \, du = \frac{1}{a}\ln|\csc(ax+b) - \cot(ax+b)| + c$

تذكر أن :

$\sec x = \frac{1}{\cos x}, \csc x = \frac{1}{\sin x}, \cot x = \frac{1}{\tan x}, \tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}$

$\sin^2 x + \cos^2 x = 1, \; 1+\tan^2 x = \sec^2 x, \; 1+\cot^2 x = coc^2 x$

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\cos 2x = 1 - 2\sin^2 x \Rightarrow \cos^2 x = \frac{1}{2}(1-\sin 2x)$

$A > B$

$\sin Ax \sin Bx = -\frac{1}{2}[\cos(Ax+Bx) - \cos(Ax-Bx)]$

$\cos Ax \cos Bx = \frac{1}{2}[\cos(Ax+Bx) + \cos(Ax-Bx)]$

$\sin Ax \cos Bx = \frac{1}{2}[\sin(Ax+Bx) + \sin(Ax-Bx)]$

$\cos Ax \sin Bx = \frac{1}{2}[\sin(Ax+Bx) - \sin(Ax-Bx)]$

أمثلة :

1) $\int \sin(3x+2)\,dx = \frac{-1}{3}\cos(3x+1) + c$

2) $\int \sec^2(1-2x)dx = \frac{-1}{2}\tan(1-2x) + c$

3) $\int \tan\left(\frac{x}{3}+5\right)dx = 3\ln\left|\sec\left(\frac{x}{3}+5\right)\right| + c$

4) $\int \csc 5x\,dx = \frac{1}{5}\ln|\csc 5x - \cot 5x| + c$

5) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \frac{1}{2\sqrt{x}}\sin\sqrt{x}\,dx = -2\cos\sqrt{x} + c$

6) $\int \frac{1}{\cos^2 2x}dx = \int \sec^2 2x\,dx = \frac{1}{2}\tan 2x + c$

7) $\int \tan^2(x+5)\,dx = \int[\sec^2(x+5) - 1]\,dx = \tan(x+5) - x + c$

8) $\int x\csc 5x^2\cot 5x^2\,dx = \frac{1}{10}\int 10x\csc 5x^2\cot 5x^2\,dx = -\frac{1}{10}\csc 5x^2 + c$

9) $\int \tan^2(2x-3)\,dx = \int[\sec^2(2x-3) - 1]dx = \frac{1}{2}\tan(2x-3) - x + C$

10) $\int \sin 3x\tan 3x\,dx = \int \frac{\sin^2 3x}{\cos 3x}dx = \int \frac{1-\cos^2 3x}{\cos 3x}dx = \int(\sec 3x + \cos 3x)\,dx$

$= \frac{1}{3}\ln|\sec 3x + \tan 3x| + \frac{1}{3}\sin 3x + C$

11) $\int \csc x\sec x\cot x\,dx = \int\left(\frac{1}{\sin x}\times\frac{1}{\cos x}\times\frac{\cos x}{\sin x}\right)dx = \int(\csc^2 x)\,dx = -\cot x + C$

12) $\int_0^{\frac{\pi}{4}} \frac{dx}{1+\cot^2 x} = \int_0^{\frac{\pi}{4}} \frac{dx}{\csc^2 x} = \int_0^{\frac{\pi}{4}} \sin^2 x\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 - \frac{1}{2}\sin 0\right)\right] = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{1}{4}\left(\frac{\pi}{2} - 1\right)$

$$13)\int (\cos x-\sin x)^2\,dx=\int (\cos^2 x-2\sin x\cos x+\sin^2 x)\,dx$$

$$=\int (1-\sin 2x)\,dx = x+\frac{1}{2}\cos 2x + C$$

$$14)\int \frac{\cos^3 x}{1-\sin x}dx=\int\left(\frac{\cos^3 x}{1-\sin x}\times\frac{1+\sin x}{1+\sin x}\right)dx=\int \frac{\cos^3 x\,(1+\sin x)}{1-\sin^2 x}dx$$

$$=\int \frac{\cos^3 x\,(1+\sin x)}{\cos^2 x}dx=\int \cos x\,(1+\sin x)\,dx=\frac{1}{2}(1+\sin x)^2+C$$

$$15)\int \frac{1}{(\cos x-\sin x)(\cos x+\sin x)}=\int \frac{1}{\cos^2 x-\sin^2 x}dx$$

$$=\int \frac{1}{\cos 2x}dx=\int \sec 2x\,dx=\frac{1}{2}\ln|\sec 2x+\tan 2x|+C$$

$$16)\int_0^{\frac{\pi}{4}} \frac{\cos 2x-1}{\cos 2x+1}\,dx=\int_0^{\frac{\pi}{4}} \frac{1-2\sin^2 x-1}{2\cos^2 x-1+1}\,dx=\int_0^{\frac{\pi}{4}} \frac{-2\sin^2 x}{2\cos^2 x}\,dx=\int_0^{\frac{\pi}{4}} -\tan^2 x\,\,dx$$

$$=\int_0^{\frac{\pi}{4}} -(\sec^2 x-1)\,dx=\int_0^{\frac{\pi}{4}} (1-\sec^2 x)\,\,dx=[x-\tan x]_0^{\frac{\pi}{4}}=\frac{\pi}{4}-1$$

$$17)\int \frac{\cot x}{\cot x+\csc x}dx=\int\left(\frac{\cot x}{\cot x+\csc x}\times\frac{\cot x-\csc x}{\cot x-\csc x}\right)dx$$

$$=\int \frac{\cot x\,(\cot x-\csc x)}{\cot^2 x-\csc^2 x}dx=\int \frac{\cot x\,(\cot x-\csc x)}{-1}dx$$

$$=\int \cot x\,(\csc x-\cot x)\,dx=\int (\cot x\csc x-\cot^2 x)\,dx$$

$$=\int (\cot x\csc x-\csc^2 x+1)\,dx=-\csc x+\cot x+x+C$$

$$18)\int \frac{\tan^2 2x}{\sec 2x}dx=\int \frac{\sec^2 2x-1}{\sec 2x}dx=\int\left(\sec 2x-\frac{1}{\sec 2x}\right)dx$$

$$=\int (\sec 2x-\cos 2x)\,dx=\frac{1}{2}\ln|\sec 2x+\tan 2x|-\frac{1}{2}\sin 2x\;+C$$

$$19)\int(\csc x-\sec x)(\sin x+\cos x)\,dx$$

$$=\int(\csc x\sin x+\csc x\cos x-\sec x\sin x-\sec x\cos x)\,dx$$

$$=\int(1+\cot x-\tan x-1)\,dx=\int(\cot x-\tan x)\,dx$$

$$=\ln|\sin x|-\ln|\sec x|+C=\ln|\sin x|+\ln|\cos x|+C=\ln|\sin x\cos x|+C$$

$$20)\int(\csc x-1)^2\,dx=\int(\csc^2 x-2\csc x+1)dx$$

$$=-\cot x-2\ln|\csc x-\cot x|+x+C$$

$$21)\int\frac{1+\sin x}{\cos^2 x}dx=\int\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx=\int(\sec^2 x+\tan x\sec x)\,dx$$

$$=\tan x+\sec x+C$$

$$22)\int\frac{1-\sin x}{\cot x}dx=\int\left(\frac{1}{\cot x}-\frac{\sin x}{\cot x}\right)dx=\int\left(\tan x-\frac{\sin^2 x}{\cos x}\right)dx$$

$$=\int\left(\tan x-\frac{1-\cos^2 x}{\cos x}\right)dx=\int(\tan x-\sec x+\cos x)\,dx$$

$$=\ln|\sec x|-\ln|\sec x+\tan x|+\sin x+C\ln\left|\frac{\sec x}{\sec x+\tan x}\right|+\sin x+C$$

$$=\ln\left|\frac{1}{1+\sin x}\right|+\sin x+C=-\ln|1+\sin x|+\sin x+C$$

$$23)\int\frac{\tan^2 x-1}{\sec^2 x}dx=\int\left(\frac{\tan^2 x}{\sec^2 x}-\frac{1}{\sec^2 x}\right)dx=\int(\sin^2 x-\cos^2 x)\,dx$$

$$=\int(-\cos 2x)\,dx=-\frac{1}{2}\sin 2x+C$$

$$24)\int sin\,3x\ \ sin\,5x\ \ dx=\int-\frac{1}{2}[cos(5x+3x)-cos(5x-3x)]\,dx$$

$$=\int-\frac{1}{2}[cos(8x)-cos(2x)]\,dx=-\frac{1}{2}\left[\frac{1}{8}sin(8x)-\frac{1}{2}\ sin\,(2x)\right]+c$$

25) $\int cos\frac{5x}{2}\ cos\frac{3x}{2}\ dx = \int \frac{1}{2}\left[cos(\frac{5x}{2}+\frac{3x}{2}) + cos(\frac{5x}{2}-\frac{3x}{2})\right] dx$

$= \int \frac{1}{2}[\cos(4x) + \cos(x)]\ dx = \frac{1}{2}\left[\frac{1}{4}\sin(4x) +\ sin\ (4x)\right] + c$

26) $\int sin\,4x\ cos\,2x\ dx = \int \frac{1}{2}[sin(4x+2x) - sin(4x-2x)]\ dx$

$= \int \frac{1}{2}[sin(6x) - sin(2x)]\ dx = \frac{1}{2}\left[-\frac{1}{6}cos(6x) - \frac{1}{4}cos(4x)\right] + c$

27) $\int cos\,2x\ sin\,x\ dx = \int \frac{1}{2}[sin(2x+x) + sin(2x-x)]\ dx$

$= \int \frac{1}{2}[sin(3x) + sin(x)]\ dx = \frac{1}{2}\left[-\frac{1}{3}cos(3x) - cos(x)\right] + c$

$\sin Ax \sin Bx = -\frac{1}{2}[\cos(Ax+Bx) - \cos(Ax-Bx)]$

$\cos Ax \cos Bx = \frac{1}{2}[\cos(Ax+Bx) + \cos(Ax-Bx)]$

$\sin Ax \cos Bx = \frac{1}{2}[\sin(Ax+Bx) + \sin(Ax-Bx)]$

$\cos Ax \sin Bx = \frac{1}{2}[\sin(Ax+Bx) - \sin(Ax-Bx)]$

تكامل الدوال الأسية

$\int a^u\,du = \frac{1}{\ln a}a^u + c$ $\int e^u\,du = e^u + c$	نتائج : $\int a^{bx+d}\ du = \frac{1}{b\ln a}a^{bx+d} + c$ $\int e^{bx+d}\ du = \frac{1}{b}\ e^{bx+d} + c$

تذكر أن :

$\log_a(xy) = \log_a x + \log_a y \quad , \quad \log_a(x/y) = \log_a x - \log_a y$

$\log_a x^n = n\log_a x \quad , \log_a a^x = x \quad , ln\, e^x = x \quad , ln\, e = 1 \quad , \log_a a = 1$

$\log_x(y) = \frac{ln\, y}{ln\, x} \quad , \log_a a^{f(x)} = f(x) \quad , a^{\log_a f(x)} = f(x) \ , e^{ln\, f(x)} = f(x)$

أمثلة :

1) $\int 3^x\,dx = \frac{1}{\ln 3}3^x + c$

2) $\int e^x\,dx = e^x + c$

3) $\int 5^{2x+1}\,dx = \frac{1}{2\ln 5}5^{2x+1} + c$

(4) $\int e^{2-3x}\,dx = -\frac{1}{3}e^{2-3x} + c$

5) $\int \frac{dx}{e^{2x}} = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + c$

6) $\int e^x(e^{x+1} - e^{-x})\,dx = \int (e^{2x+1} - 1)\,dx = \frac{1}{2}e^{2x+1} - \ x + c$

7) $\int \frac{e^{4x+1} - e^{2x+4}}{e^{2x+3}}dx = \int \left(\frac{e^{5x+1}}{e^{2x+3}} - \frac{e^{2x+4}}{e^{2x+3}}\right)dx = \int (e^{3x-2} - e)\,dx$

$= \frac{1}{3}e^{3x-2} + e\,x + c$

8) $\int \frac{(e^{2x} + e^{3x})^2}{e^{5x}}dx = \int \frac{e^{4x} + 2e^{5x} + e^{6x}}{e^{5x}}dx = \int (e^{-x} + 2 + e^x)\,dx$

$= -e^{-x} + 2x + e^x + C$

9) $\int e^{3\ln(x+1)}\,dx = \int e^{\ln(x+1)^3}\,dx = \int (x+1)^3\,dx = \frac{(x+1)^4}{4} + c$

10) $\int 5^{2\log_5(x-2)}\,dx = \int 5^{\log_5(x-2)^2}\,dx = \int (x-2)^2\,dx = \dfrac{(x-2)^3}{3} + c$

11) $\int 2x\,e^{x^2}\,dx = e^{x^2} + c$

12) $\int \cos x \;\; e^{\sin x}\,dx = e^{\sin x} + c$

13) $\int e^x\,e^{e^x}\,dx = e^{e^x} + c$

14) $\int 2^x\,e^{2^x}\,dx = \ln 2 \int \dfrac{2^x}{\ln 2}\,e^{2^x}\,dx = (\ln 2)\;\; e^{2^x} + c$

15) $\int \dfrac{e^{\tan x}}{\cos^2 x}\,dx = \int \sec^2 x \;\; e^{\tan x}\,dx = e^{\tan x} + c$

16) $\int_{\ln 2}^{\ln 4} e^{2x}\;\,dx = \dfrac{1}{2}\left[e^{2x}\right]_{\ln 2}^{\ln 4} = \dfrac{1}{2}\left[e^{2\ln 4} - e^{2\ln 2}\right] = \dfrac{1}{2}\left[e^{\ln 16} - e^{\ln 4}\right]$

$= \dfrac{1}{2}[16 - 4] = 6$

17) $\int_{\ln\log_4 \ln 4}^{\ln\log_4 \ln 16} e^x\,4^{e^x}\,dx = \dfrac{1}{\ln 4}\left[4^{e^x}\right]_{\ln\log_4 \ln 4}^{\ln\log_4 \ln 16}$

$= \dfrac{1}{\ln 4}\left[4^{e^{\ln\log_4 \ln 16}} - 4^{e^{\ln\log_4 \ln 16}}\right] = \dfrac{1}{\ln 4}\;\left[4^{\log_4 \ln 16} - 4^{\log_4 \ln 4}\right]$

$= \dfrac{1}{\ln 4}\;[\ln 16 - \ln 4] = \dfrac{1}{\ln 4}\;[\ln 4] = 1$

18) $\int_1^4 \dfrac{1}{\sqrt{x}\;\,e^{\sqrt{x}}}\,dx = \int_1^4 \dfrac{1}{\sqrt{x}}\;\,e^{-\sqrt{x}}\,dx = -2\int_1^4 \dfrac{-1}{2\sqrt{x}}\;\,e^{-\sqrt{x}}\,dx$

$= -2\left[e^{-\sqrt{x}}\right]_1^4 = -2\left[e^{-2} - e^{-1}\right] = 2\left[e^{-1} - e^{-2}\right]$

19) $\int \dfrac{\sec^3 x + \;e^{\sin x}}{\sec x}\,dx = \int \sec^2 x + \cos x\,e^{\sin x}\,dx = \tan x + e^{\sin x} + C$

تكامل الدوال المثلثية العكسية

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)+c$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right)+c$$

أمثلة /

1) $$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{(3)^2-x^2}} = \sin^{-1}\frac{x}{3}+c$$

2) $$\int \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\int \frac{3\,dx}{\sqrt{(4)^2-(3x)^2}} = \frac{1}{3}\sin^{-1}\frac{3x}{4}+c$$

3) $$\int \frac{-\,dx}{\sqrt{2-3x^2}} = \frac{1}{\sqrt{3}}\int \frac{-\sqrt{3}\,dx}{\sqrt{\left(\sqrt{2}\right)^2-\left(\sqrt{3}\,x\right)^2}} = \frac{1}{\sqrt{3}}\cos^{-1}\frac{\sqrt{3}\,x}{\sqrt{2}}+c$$

4) $$\int \frac{dx}{\sqrt{x}\,\sqrt{4-x}} = \int \frac{dx}{\sqrt{x}\,\sqrt{(2)^2-\left(\sqrt{x}\right)^2}} = 2\int \frac{dx}{2\sqrt{x}\,\sqrt{(2)^2-\left(\sqrt{x}\right)^2}}$$

$$= 2\sin^{-1}\frac{\sqrt{x}}{2}+c$$

5) $$\int \frac{\sec^2 x\,dx}{\sqrt{25-\tan^2 x}} = \int \frac{\sec^2 x\,dx}{\sqrt{(5)^2-(\tan x)^2}} = \sin^{-1}\left(\frac{\tan x}{5}\right)+c$$

6) $$\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int_0^{3\sqrt{2}/4} \frac{2\,dx}{\sqrt{(3)^2-(2x)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)\Big|_0^{3\sqrt{2}/4}$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{2\left(3\sqrt{2}/4\right)}{3}\right)-\sin^{-1}\left(\frac{0}{3}\right)\right] = \frac{1}{2}\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)-\sin^{-1}(0)\right] = \frac{\pi}{8}$$

$$7)\ \int \frac{x^2\,dx}{\sqrt{4-x^6}} = \frac{1}{3}\int \frac{3x^2\,dx}{\sqrt{(2)^2-(x^3)^2}} = \frac{1}{3}\sin^{-1}\frac{x^3}{2} + c$$

$$8)\ \int \frac{dx}{4+x^2} = \int \frac{dx}{(2)^2+(x)^2} = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

$$9)\ \int \frac{x\,dx}{3+x^4} = \frac{1}{2}\int \frac{2x\,dx}{(\sqrt{3})+(x^2)^2} = \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x^2}{\sqrt{3}} + c$$

$$10)\int \frac{e^{2x}\,dx}{1+e^{4x}} = \frac{1}{2}\int \frac{2e^{2x}\,dx}{1+(e^{2x})^2} = \frac{1}{2}\tan^{-1}\frac{e^{2x}}{1} + c = \frac{1}{2}\tan^{-1}(e^{2x}) + c$$

$$11)\ \int \frac{(1+x)dx}{4+x^2} = \int\left(\frac{1}{4+x^2}+\frac{x}{4+x^2}\right)dx = \frac{1}{2}\tan^{-1}\frac{x}{2}+\frac{1}{2}\ln(4+x^2)+c$$

$$12)\ \int_0^{\pi/2} \frac{\sin x\,dx}{1+\cos^2 x} = -\int_0^{\pi/2} \frac{-\sin x\,dx}{1+(\cos x)^2} = \tan^{-1}(\cos x)\Big|_0^{\frac{\pi}{2}}$$

$$= \tan^{-1}(\cos\pi/2) - \tan^{-1}(\cos 0) = \tan^{-1}(0) - \tan^{-1}(1) = -\frac{\pi}{4}$$

$$13)\ \int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{dx}{x\sqrt{(x)^2-(2)^2}} = \frac{1}{2}\sec^{-1}\frac{x}{2} + c$$

$$14)\ \int \frac{dx}{x\sqrt{3x^2-7}} = \sqrt{3}\int \frac{dx}{\sqrt{3}\,x\sqrt{\left(\sqrt{3}\,x\right)^2-\left(\sqrt{7}\right)^2}} = \frac{\sqrt{3}}{\sqrt{7}}\sec^{-1}\frac{\sqrt{3}\,x}{\sqrt{7}} + c$$

$$15)\int \frac{dx}{\sqrt{e^{2x}-4}} = \int \frac{e^x\,dx}{e^x\sqrt{(e^x)^2-(2)^2}} = \frac{1}{2}\sec^{-1}\left(\frac{e^x}{2}\right) + c$$

$$16)\ \int \frac{dx}{x\sqrt{(\ln x)^2-25}} = \int \frac{dx}{x\sqrt{(\ln x)^2-(5)^2}} = \frac{1}{5}\sec^{-1}\left(\frac{\ln x}{5}\right) + c$$

$$17)\ \int \frac{dx}{x\sqrt{x^4-5}} = \int \frac{x\,dx}{x^2\sqrt{(x^2)^2-\left(\sqrt{5}\right)^2}} = \frac{1}{2}\int \frac{2x\,dx}{x^2\sqrt{(x^2)^2-\left(\sqrt{5}\right)^2}}$$

$$= \frac{1}{2\sqrt{5}}\sec^{-1}\frac{x^2}{2} + c$$

$$18)\int \frac{dx}{4x+x(\ln x)^2}=\int \frac{dx}{x\,(4+(\ln x)^2)}=\int \frac{\frac{1}{x}\,dx}{4+(\ln x)^2}=\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{2}\right)$$

$$19)\int \frac{dx}{\sqrt{4-(3x+1)^2}}=\frac{1}{3}\int \frac{3dx}{\sqrt{(2)^2-(3x+1)^2}}=\sin^{-1}\left(\frac{3x+1}{2}\right)+c$$

$$20)\int \frac{dx}{4+(2-x)^2}=-\int \frac{-\,dx}{(2)^2+(2-x)^2}=\frac{1}{2}\cot^{-1}\left(\frac{2-x}{2}\right)+c$$

$$21)\int \frac{dx}{(2x+3)\sqrt{(2x+3)^2-25}}=\int \frac{2\,dx}{(2x+3)\sqrt{(2x+3)^2-(5)^2}}$$

$$=\frac{1}{5}\sec^{-1}\left(\frac{2x+3}{5}\right)+c$$

$$22)\int \frac{dx}{\sqrt{4x-3-x^2}}=\int \frac{dx}{\sqrt{1-(x+2)^2}}$$

$$=\sin^{-1}\left(\frac{x+2}{1}\right)+c$$

$$=\sin^{-1}(x+2)+c$$

إكمال مربع : $(-x^2+4x-3)$

$$=-(x^2-4x+3)$$
$$=-[x^2+4x+4-4+3]$$
$$=-[(x+2)^2-1]$$
$$=1-(x+2)^2$$

$$23)\int \frac{dx}{x^2-2x+5}=\int \frac{dx}{(2)^2+(x-1)^2}$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c$$

$$x^2-2x+5$$
$$=x^2-2x+1-1+5$$
$$=(x-1)^2+4$$
$$=4+(x-1)^2$$
$$=2^2+(x-1)^2$$

$$24)\int \frac{dx}{(x+1)\sqrt{x^2+2x}}=\int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}$$

$$=\sec^{-1}\left(\frac{x+1}{1}\right)+c=\sec^{-1}(x+1)+c$$

$$x^2+2x$$
$$=x^2+2x+1-1$$
$$=(x+1)^2-1$$

$$25)\int \frac{\sin x\ \cos x}{\sin^4 x+\cos^4 x}dx$$

$$=\int \frac{\frac{\sin x\ \cos x}{\cos^4 x}}{\frac{\sin^4 x}{\cos^4 x}+1}dx=\int \frac{\tan x\ \sec^2 x}{\tan^4 x+1}dx$$

$$=\int \frac{\tan x\ \sec^2 x}{(\ \tan^2 x)^2+1}dx=\frac{1}{2}\int \frac{2\tan x\ \sec^2 x}{(\ \tan^2 x)^2+1}dx$$

$$=\frac{1}{2}\tan^{-1}(\ \tan^2 x)+c$$

$$26)\int \frac{\cos x\ \ dx}{\sqrt{3+\cos 2x}}=\int \frac{\cos x\ \ dx}{\sqrt{3+1-2\sin^2 x}}=\int \frac{\cos x\ \ dx}{\sqrt{4-2\sin^2 x}}$$

$$=\int \frac{\cos x\ \ dx}{\sqrt{(2)^2-\left(\sqrt{2}\sin x\right)^2}}=\frac{1}{\sqrt{2}}\int \frac{\sqrt{2}\ \cos x\ \ dx}{\sqrt{(2)^2-\left(\sqrt{2}\sin x\right)^2}}$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}\sin x}{2}\right)+c$$

تكامل الدوال الزائدية

$\int \cosh u \; du = \sinh u + c$	$\int \operatorname{csch}^2 u \; du = -\coth u + c$
$\int \sinh u \; du = \cosh u + c$	$\int \operatorname{sech} u \tanh u \; du = -\operatorname{sech} u + c$
$\int \operatorname{sech}^2 u \; du = \tanh u + c$	$\int \operatorname{csch} u \coth u \; du = -\operatorname{csch} u + c$

أمثلة /

1) $\int \cosh 3x \; dx = \frac{1}{3}\int 3 \cosh 3x \; dx = \frac{1}{3} \sinh 3x + c$

2) $\int \frac{1}{x}\sinh(\ln x) \; dx = \frac{1}{3} \sinh(\ln x) + c$

3) $\int \frac{\tanh\sqrt{x}}{\sqrt{x}} \; dx = \int \frac{\sinh\sqrt{x}}{\sqrt{x} \; \cosh\sqrt{x}} \; dx = 2\int \frac{\frac{1}{2\sqrt{x}} \; \sinh\sqrt{x}}{\cosh\sqrt{x}} \; dx = \ln\left| \cosh\sqrt{x}\right| + c$

4) $\int_0^{\ln 2} 4\, e^{x} \; \sinh x \; dx = \int_0^{\ln 2} 4\, e^{x}\left(\frac{e^{x} - e^{-x}}{2}\right) \; dx = \int_0^{\ln 2} (2\, e^{2x} - 2) \; dx$

$= [e^{2x} - 2x]_0^{\ln 2} = \left[e^{2\ln 2} - 2\ln 2\right] - [1 - 0] = 4 - 2\ln 2 - 1 = 3 - 2\ln 2$

5) $\int \frac{1}{x^2}\operatorname{sech}\frac{1}{x} \; \tanh\frac{1}{x} \; dx = -\int \frac{-1}{x^2}\operatorname{sech}\frac{1}{x} \; \tanh\frac{1}{x} \; dx = \operatorname{sech}\frac{1}{x} + c$

6) $\int x \; \operatorname{sech}^2(3 - x^2) \; dx = -\frac{1}{2}\int (-2x) \; \operatorname{sech}^2(3 - x^2)dx = \tanh(3 - x^2) + c$

7) $\int \cos x \operatorname{csch}(\sin x) \; \coth(\sin x) \; du = -\operatorname{csch}(\sin x) + c$

8) $\int \cosh^2 2x \; dx = \int \frac{\cosh 4x + 1}{2} \; dx = \frac{1}{2}\int (\cosh 4x + 1) \; dx = \frac{1}{2}\left[\frac{\sinh 4x}{4} + x\right] + c$

$$9)\int_0^1 \sinh^2 x\ dx=\int_0^1 \frac{\cosh 2x-1}{2}\ dx=\frac{1}{2}\int_0^1 (\cosh 2x-1)\ dx=\frac{1}{2}\left[\frac{\sinh 2x}{2}-x\right]_0^1$$

$$=\frac{1}{2}\left[\frac{\sinh 2}{2}-1\right]$$

$$10)\int \sinh^5 x\cosh x\ dx=\frac{\sinh^6 x}{6}+c$$

$$11)\int \sqrt{\tanh x}\,\mathrm{sech}^2\, x\ dx=\int (\tanh x)^{1/2}\,\mathrm{sech}^2\, x\ dx=\frac{(\tanh x)^{3/2}}{3/2}+c$$

$$12)\int \sinh^5 x\ dx=\int \sinh^4 x\ \ \sinh x\ dx=\int (\cosh^2 x-1)^2\ \ \sinh x\ dx$$

$$=\int (\cosh^4 x-2\cosh^2 x+\ 1)\ \ \sinh x\ dx$$

$$=\int (\cosh^4 x\ \ \sinh x-2\cosh^2 x\ \ \sinh x+\ \sinh x)\ \ \ dx$$

$$=\frac{\cosh^5 x}{5}-\frac{2\cosh^3 x}{3}+\cosh x+c$$

$$13)\int \coth^2 x\ dx=\int (1+\mathrm{csch}^2\, x)\,dx=x-\coth x+c$$

$$14)\int \tanh^3 x\ \ dx=\int \tanh^2 x\ \tanh x\ dx=\int (1-\mathrm{sech}^2\, x)\ \tanh x\ dx$$

$$=\int (\tanh x-\mathrm{sech}^2\, x\ \tanh x)\ dx=\ln|\cosh x|+\frac{\tanh^2 x}{2}+c$$

$$15)\int \mathrm{sech}^4\, x\ dx=\int \mathrm{sech}^2\, x\ \mathrm{sech}^2\, x\ dx=\int \mathrm{sech}^2\, x\ \ (1-\tanh^2 x)\,dx=$$

$$=\int (\mathrm{sech}^2\, x-\tanh^2 x\ \ \ \mathrm{sech}^2\, x)\ \ dx=\tanh x-\frac{\tanh^3 x}{3}+c$$

تكامل الدوال الزائدية العكسية

$\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + c \quad , \ a>0$	$\sinh^{-1}u = \ln(u+\sqrt{u^2+1}) , -\infty < u < \infty$
$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + c \quad ,u>a>0$	$\cosh^{-1}u = \ln(u+\sqrt{u^2-1}) \quad ,u \geq 1$
$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)+c & ,\lvert u\rvert < a \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right)+c & ,\lvert u\rvert > a \end{cases}$	$\tanh^{-1}u = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) \quad ,\lvert u\rvert<1$
$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + c \ ,0>u>a$	$\operatorname{sech}^{-1}u = \ln\left(\frac{1+\sqrt{1-u^2}}{u}\right) \ ,0<u\leq 1$
$\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left(\frac{u}{a}\right) + c \ ,u \neq 0$	$\operatorname{csch}^{-1}u = \ln\left(\frac{1}{u}+\frac{\sqrt{1+u^2}}{\lvert u\rvert}\right) \ ,u\neq 0$
	$\tanh^{-1}u = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) \quad ,\lvert u\rvert>1$

مثال /

1) $\int \frac{dx}{\sqrt{1+9x^2}} = \int \frac{dx}{\sqrt{1+(3x)^2}} = \sinh^{-1}\left(\frac{3x}{1}\right) + c = \sinh^{-1}(3x) + c$

2) $\int \frac{dx}{\sqrt{9x^2-2}} = \int \frac{dx}{\sqrt{(3x)^2-(\sqrt{2})^2}} = \cosh^{-1}\left(\frac{3x}{\sqrt{2}}\right) + c$

3) $\int \frac{dx}{\sqrt{4-25e^{2x}}} = \int \frac{5e^x\,dx}{5e^x\sqrt{(2)^2-(5e^x)^2}} = -\frac{1}{2}\operatorname{sech}^{-1}\left(\frac{5e^x}{2}\right) + c$

4) $\int \frac{dx}{\sqrt{x}\sqrt{1+x}} = 2\int \frac{dx}{2\sqrt{x}\sqrt{1+(\sqrt{x})^2}} = -2\operatorname{csch}^{-1}\left(\frac{\sqrt{x}}{1}\right) + c = 2\operatorname{csch}^{-1}(\sqrt{x}) + c$

$$5)\int \frac{dx}{1-4x-2x^2}$$

$$1-4x-2x^2=-2[x^2+2x-\frac{1}{2}]$$

$$=-2[x^2+2x+1-1-\frac{1}{2}]$$

$$=-2\left[(x+1)^2-\frac{3}{2}\right]=2\left[\frac{3}{2}-(x+1)^2\right]$$

$$=\int \frac{dx}{2[2-(x+1)^2]}=\frac{1}{2}\int \frac{dx}{[2-(x+1)^2]}$$

$$=\int \frac{dx}{2\left[\frac{3}{2}-(x+1)^2\right]}=\frac{1}{2}\int \frac{dx}{\left[\frac{3}{2}-(x+1)^2\right]}$$

$$=\frac{1}{2}\cdot\frac{\sqrt{2}}{\sqrt{3}}\coth^{-1}\left(\frac{\sqrt{2}(x+1)}{\sqrt{3}}\right)+c=\frac{1}{\sqrt{6}}\tanh^{-1}\left(\frac{\sqrt{2}(x+1)}{\sqrt{3}}\right)+c$$

$$6)\int \frac{dx}{(x+1)\sqrt{2x^2+4x+8}}=\frac{1}{\sqrt{2}}\int \frac{dx}{(x+1)\sqrt{x^2+2x+4}}$$

$$=\frac{1}{\sqrt{2}}\int \frac{dx}{(x+1)\sqrt{x^2+2x+1+1+3}}=\frac{1}{\sqrt{2}}\int \frac{dx}{(x+1)\sqrt{(x+1)^2+3}}$$

$$=-\frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\operatorname{csch}^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c=-\frac{1}{\sqrt{6}}\operatorname{csch}^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c$$

$$7)\int \frac{\sqrt{x}\ \ dx}{\sqrt{1+x^3}}=\frac{2}{3}\int \frac{(3/2)x^{1/2}\ dx}{\sqrt{1+(x^{3/2})^2}}=\frac{2}{3}\ \sinh^{-1}\left(\frac{x^{3/2}}{1}\right)+c\ =\frac{2}{3}\ \sinh^{-1}(x^{3/2})+c$$

$$8)\int_3^7 \frac{dx}{\sqrt{x^2-4}}=\left[\cosh^{-1}\left(\frac{x}{2}\right)\right]_3^7\ =\left[\ln\left(\frac{x}{2}+\sqrt{\frac{x^2}{4}-1}\right)\right]_3^7$$

$$=\left[\ln\left(\frac{7}{2}+\sqrt{\frac{49}{4}-1}\right)\right]-\left[\ln\left(\frac{3}{2}+\sqrt{\frac{9}{4}-1}\right)\right]=\left[\ln\left(\frac{7}{2}+\sqrt{\frac{45}{4}}\right)\right]-\left[\ln\left(\frac{3}{2}+\sqrt{\frac{5}{4}}\right)\right]$$

$$=\ln\left(\frac{7}{2}+\frac{3\sqrt{5}}{2}\right)-\ln\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)=\ln\left(\frac{7+3\sqrt{5}}{2}\right)-\ln\left(\frac{3+\sqrt{5}}{2}\right)$$

$$=\ln\left(\frac{7+3\sqrt{5}}{3+\sqrt{5}}\right)=\ln\left(\frac{3+\sqrt{5}}{2}\right)$$

القواعد الأساسية للتكامل

4. $\int [f(x)]^n f'(x) \; dx = \frac{[f(x)]^n}{n+1} + C \quad , n \neq -1$

5. $\int \frac{f'(x)}{f(x)} \; dx = \ln|f(x)| + C$

نتائج :

* $\int (ax+b)^n \; dx = \frac{1}{a}\frac{(ax+b)^n}{n+1} + C \quad , n \neq -1$

* $\int (ax+b)^{-1} \; dx = \frac{1}{a}\ln|ax+b| + C$

أمثلة:

1) $\int (3x+4)^8 \, dx = \frac{1}{3}\frac{(3x+4)^9}{9} + C = \frac{1}{27}(3x+4)^9 + C$

2) $\int (4-5x)^3 \, dx = -\frac{1}{5}\frac{(4-5x)^4}{4} + C = -\frac{1}{20}(4-5x)^4 + C$

3) $\int \frac{2}{\sqrt[3]{5x-1}} \, dx = \int (5x-1)^{-\frac{1}{3}} \, dx = 2 \times \frac{1}{5}\frac{3(5x-1)^{\frac{2}{3}}}{2} + C = \frac{3}{5}(5x-1)^{\frac{4}{3}} + C$

4) $\int \frac{2}{2x+3} \, dx = \ln(2x+3) + C$

(5) $\int \frac{1}{2-5x} \, dx = -\frac{1}{5}\ln(2-5x) + C$

6) $\int 3x^2 (x^3+4)^8 \, dx = \frac{(x^3+4)^9}{9} + C$

7) $\int \frac{3x^2}{x^3+4} \, dx = \ln|x^3+4| + C$

8) $\int \cos x \sqrt{1+\sin x} \, dx = \int \cos x (1+\sin x)^{\frac{1}{2}} \, dx = \frac{2(1+\sin x)^{\frac{3}{2}}}{3} + C$

9) $\int \frac{\cos x}{1+\sin x} \, dx = \ln|1+\sin x| + C$

10) $\int 2e^{2x}(e^{2x}-3)^8 \, dx = \frac{(e^{2x}-3)^9}{9} + C$

$$9)\int_0^{1/2}\frac{dx}{1-x^2}=[\tanh^{-1}x]_0^{1/2}=\tanh^{-1}\frac{1}{2}-\tanh^{-1}0=\tanh^{-1}\frac{1}{2}$$

$$=\frac{1}{2}\ln\left(\frac{1+1/2}{1-1/2}\right)=\frac{1}{2}\ln(3)$$

$$\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\coth^{-1}x=\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$10)\int_{5/4}^{2}\frac{dx}{1-x^2}=[\coth^{-1}x]_{5/4}^{2}=\coth^{-1}2-\coth^{-1}\frac{5}{4}$$

$$=\frac{1}{2}\ln\left(\frac{1+2}{2-1}\right)-\frac{1}{2}\ln\left(\frac{5/4+1}{5/4-1}\right)=\frac{1}{2}\ln(3)-\frac{1}{2}\ln(9)=\frac{1}{2}\ln(3/9)=\frac{1}{2}\ln\left(\frac{1}{3}\right)$$

$$11)\int_1^2\frac{dx}{x\sqrt{4+x^2}}=-\frac{1}{2}\left[\operatorname{csch}^{-1}\frac{x}{2}\right]_1^2=-\frac{1}{2}\left(\operatorname{csch}^{-1}1-\operatorname{csch}^{-1}\frac{1}{2}\right)$$

$$=-\frac{1}{2}\left[\ln(2+\frac{\sqrt{5/4}}{1/2})-\ln\left(1+\sqrt{2}\right)\right]$$

$$\operatorname{csch}^{-1}x=\ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right)$$

$$=-\frac{1}{2}\left[\ln(2+\sqrt{5})-\ln\left(1+\sqrt{2}\right)\right]\quad=-\frac{1}{2}\left[\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)\right]$$

11) $\int \frac{2\,e^{2x}}{e^{2x}-3}\,dx = \ln|e^{2x}-3| + C$

12) $\int x\sec^2 x^2\ (\tan x^2+4)^3\,dx = \frac{1}{2}\int 2x\sec^2 x^2\ (\tan x^2+4)^3\,dx$

$$= \frac{1}{2}\ \frac{(\tan x^2+4)^4}{4} + C = \frac{1}{8}\ (\tan x^2+4)^4 + C$$

13) $\int \frac{x-1}{x^2-2x+3}\,dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x+3}\,dx = \ln|x^2-2x+3| + C$

14) $\int \cos 3x\ \sinh(\sin 3x)\,[3-\cosh(\sin 3x)]^4\,dx$

$$= -\frac{1}{3}\int -3\cos 3x\ \sinh(\sin 3x)\,[3-\cosh(\sin 3x)]^4\,dx$$

$$= -\frac{1}{3}\frac{[3-\cosh(\sin 3x)]^5}{5} = -\frac{1}{15}[3-\cosh(\sin 3x)]^5 + C$$

15) $\int \frac{e^{\sin^{-1}x}\,dx}{\sqrt{1-x^2}} = \int e^{\sin^{-1}x}\ \frac{1\,dx}{\sqrt{1-x^2}} = e^{\sin^{-1}x} + c$

16) $\int \frac{(\sin^{-1}x)^2\,dx}{\sqrt{1-x^2}} = \int (\sin^{-1}x)^2\cdot\frac{dx}{\sqrt{1-x^2}} = \frac{(\sin^{-1}x)^3}{3} + c$

17) $\int \frac{\sqrt{\tan^{-1}x}\ dx}{1+x^2} = \int (\tan^{-1}x)^{1/2}\cdot\frac{dx}{1+x^2} = \frac{(\tan^{-1}x)^{3/2}}{3/2} + c$ x^2

18) $\int \frac{dx}{(\sin^{-1}x)\sqrt{1-x^2}} = \int \frac{\frac{1}{\sqrt{1-x^2}}\,dx}{(\sin^{-1}x)} = \ln|\sin^{-1}x| + c$

19) $\int \frac{dx}{(\tan^{-1}x)\,(1+x^2)} = \int \frac{\frac{1}{1+x^2}\,dx}{(\tan^{-1}x)} = \ln|\tan^{-1}x| + c$

20) $\int (x^2-6x+9)^{\frac{5}{2}}\,dx = \int [(x-3)^2]^{\frac{5}{2}}\,dx = \int (x-3)^5\,dx = \frac{(x-3)^6}{6} + C$

$\sqrt{x^2}$ $\sqrt{9}$

x $2*$ 3

$6x$

$$21)\ \int \frac{x}{x^4-6x^2+9}\,dx = \int \frac{x}{(x^2-3)^2}\,dx = \int x(x^2-3)^{-2}\,dx$$

$$= \frac{1}{2}\int 2x(x^2-3)^{-2}\,dx = -\frac{1}{2}(x^2-3)^{-1} + C$$

$$22)\ \int \sqrt{x^2-2x^4}\,dx = \int \sqrt{x^2(1-2x^2)}\,dx = \int x\,(1-2x^2)^{\frac{1}{2}}\,dx$$

$$= -\frac{1}{4}\int -4x\,(1-2x^2)^{\frac{1}{2}}\,dx = -\frac{1}{4}\times\frac{2(1-2x^2)^{\frac{3}{2}}}{3} + C = -\frac{1}{6}(1-2x^2)^{\frac{3}{2}} + C$$

$$23)\ \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx = \int \frac{1}{x^2}\left(1-\frac{1}{x}\right)^{\frac{1}{2}}\,dx = \frac{2}{3}\left(1-\frac{1}{x}\right)^{\frac{3}{2}} + C$$

$$24)\ \int \sqrt{\frac{1+\sqrt{x}}{x}}\,dx = \int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}}\,(1+\sqrt{x})^{\frac{1}{2}}\,dx$$

$$= 2\int \frac{1}{2\sqrt{x}}\,(1+\sqrt{x})^{\frac{1}{2}}\,dx = 2\,\frac{2(1+\sqrt{x})^{\frac{3}{2}}}{3} + C = \frac{4}{3}\,(1+\sqrt{x})^{\frac{3}{2}} + C$$

$$25)\ \int \frac{x}{x-1}\,dx = \int \frac{x-1+1}{x-1}\,dx = \int \left(1+\frac{1}{x-1}\right)dx = x + \ln|x-1|$$

$$26)\ \int \frac{x^2-3}{x^2+4}\,dx = \int \frac{x^2+4-7}{x^2+4}\,dx = \int \left(1-\frac{7}{x^2+4}\right)dx$$

$$= x - 7\times\frac{1}{2}\tan^{-1}\frac{x}{2} + C = x - \frac{7}{2}\tan^{-1}\frac{x}{2} + C$$

$$27)\ \int \frac{x^2-3}{x+3}\,dx = \int \left(x-3+\frac{6}{x+3}\right)dx$$

$$= \frac{1}{2}\,x^2 - 3x + 6\ln|x+3| + C$$

$$\begin{array}{r|l} & x-3 \\ \hline x+3 & x^2+0x-3 \\ & x^2+3x \\ \hline & 0-3x-3 \\ & -3x-9 \\ \hline & +6 \end{array}$$

$$28)\ \int \frac{x^2+2x}{(x+1)^2}\,dx = \int \frac{x^2+2x}{x^2+2x+1}\,dx = \int \frac{x^2+2x+1-1}{x^2+2x+1}\,dx$$

$$= \int \left(1-\frac{1}{x^2+2x+1}\right)dx = \int \left(1-\frac{1}{(x+1)^2}\right)dx = \int (1-(x+1)^{-2})\,dx$$

$$= x - (x+1)^{-1} + C$$

29) $\int \frac{\sqrt{\ln x}}{x}dx = \int \frac{1}{x}(\ln x)^{\frac{1}{2}}dx = \frac{2}{3}(\ln x)^{\frac{3}{2}} + C$

30) $\int \frac{1}{x\sqrt{1-\ln x}}dx = \int \frac{1}{x}(1-\ln x)^{-\frac{1}{2}}dx = -\int -\frac{1}{x}(1-\ln x)^{-\frac{1}{2}}dx$

$$= -2(1-\ln x)^{\frac{1}{2}} + C = -2\sqrt{1-\ln x} + C$$

31) $\int \frac{1}{x\ln x}dx = \int \frac{\frac{1}{x}}{\ln x}dx = \ln|\ln x| + C$

32) $\int \frac{1}{x(3+2\ln x)}dx = \int \frac{\frac{1}{x}}{3+2\ln x}dx = \frac{1}{2}\int \frac{\frac{2}{x}}{3+2\ln x}dx = \frac{1}{2}\ln|3+2\ln x| + C$

33) $\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx = \frac{1}{2}\int \frac{2(e^{2x}+e^{-2x})}{e^{2x}-e^{-2x}}dx = \frac{1}{2}\ln(e^{2x}-e^{-2x}) + C$

34) $\int \frac{2^x+2^{-x}}{2^x-2^{-x}}dx = \frac{1}{\ln 2}\int \frac{(2^x+2^{-x})\ln 2}{2^x-2^{-x}}dx = \frac{1}{\ln 2}\ln(2^x-2^{-x}) + C$

35) $\int \frac{1}{3-2e^{2x}}dx = \int \frac{e^{-2x}}{3e^{-2x}-2}dx = -\frac{1}{6}\int \frac{-6\,e^{-2x}}{3e^{-2x}-2}dx = -\frac{1}{6}\ln(3e^{-2x}-2) + C$

36) $\int \frac{e^x+1}{e^x-2}\,dx = \int \left(\frac{e^x}{e^x-2} + \frac{1}{e^x-2}\right)dx = \int \frac{e^x}{e^x-2}dx + \int \frac{1}{e^x-2}dx$

$$= \int \frac{e^x}{e^x-2}dx + \int \frac{e^{-x}}{1-2e^{-x}}dx = \int \frac{e^x}{e^x-2}dx - \frac{1}{2}\int \frac{-2e^{-x}}{1-2e^{-x}}dx$$

$$= \ln(e^x-2) - \frac{1}{2}\ln(1-2e^{-x}) + C$$

37) $\int \frac{e^{-x}}{e^{-x}-e^x}dx = \int \frac{e^{-2x}}{e^{-2x}-1}dx = -\frac{1}{2}\int \frac{-2e^{-2x}}{e^{-2x}-1}dx = -\frac{1}{2}\ln|e^{-2x}-1| + C$

38) $\int \left(\frac{\sec x}{1+\tan x}\right)^2 dx = \int \frac{\sec^2 x}{(1+\tan x)^2}\,dx = \int \sec^2 x\,(1+\tan x)^{-2}\,dx$

$$= \frac{(1+\tan x)^{-1}}{-1} + C = \frac{-1}{1+\tan x} + C$$

التكامل بالتعويض :

نواجه أحياناً تكاملات على الصورة $\int f(g(x))\, g'(x)\;\; dx$ ولا يمكن إيجادها باستخدام القواعد الأساسية مباشرة

فإذا فرضنا $u = g(x) \;\; \Leftarrow \;\; du = g'(x)\, dx$

ويصبح لدينا :

$$\int f(g(x))\, g'(x)\;\; dx = \int f(u)\;\, du$$

أمثلة :

1) $\int x(x^2+4)^{10}\, dx$ $\qquad u = x^2 + 4 \Rightarrow du = 2x\, dx$

$$= \frac{1}{2}\int (x^2+4)^{10}\,(2x\, dx) = \int u^{10}\, du = \frac{1}{2}\,\frac{u^{11}}{11} + c = \frac{(x^2+4)^{11}}{22} + c$$

2) $\int \frac{\sqrt{\sqrt{x}+2}}{\sqrt{x}}\, dx$ $\qquad u = \sqrt{x} + 2 \Rightarrow du = \frac{1}{2\sqrt{x}}\, dx$

$$= 2\int \left(\sqrt{x}+2\right)^{1/2} \frac{1}{2\sqrt{x}}\, dx = 2\int (u)^{1/2}\;\, du = 2\,\frac{u^{1/2}}{1/2} + c = \left(\sqrt{x}+2\right)^{1/2} + c$$

3) $\int \frac{dx}{x + 2x\ln x} = \int \frac{dx}{x(1+2\ln x)}$ $\qquad u = 1 + 2\ln x \Rightarrow du = \frac{2}{x}\, dx$

$$= \frac{1}{2}\int \frac{2dx/x}{(1+2\ln x)} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|1+2\ln x| + c$$

4) $\int x^2\sqrt{x+3}\;\; dx = \int x^2(x+3)^{1/2}\;\; dx$ $\qquad u = x + 3 \Rightarrow du = \; dx$

$x = u - 3$

$$= \int (u-3)^2\, u^{\frac{1}{2}}\;\, du = \int (u^2 - 6u + 9)\, u^{\frac{1}{2}}\;\, du$$

$$= \int \left(u^{5/2} - 6u^{3/2} + u^{1/2}9\right)\quad du = \frac{u^{7/2}}{7/2} - \frac{6\,u^{5/2}}{5/2} + \frac{9\,u^{3/2}}{3/2} + c$$

$$= \frac{2\,(x+3)^{7/2}}{7} - \frac{12\,(x+3)^{5/2}}{5} + 2\,(x+3)^{3/2} + c$$

5) $\int \sin^3 x \sqrt{\cos x}\, dx = \int (1-\cos^2 x)\sqrt{\cos x}\ \sin x\, dx$ $\qquad u=\cos x \Rightarrow du = -\sin x\, dx$

$\Rightarrow -du = \sin x\, dx$

$$= -\int (1-u^2)\sqrt{u}\ du = -\int \left(u^{1/2} - u^{5/2}\right)\ du$$

$$= \int \left(u^{5/2} - u^{1/2}\right)\ du = \frac{u^{7/2}}{7/2} - \frac{u^{3/2}}{3/2} + c = \frac{2(\cos x)^{7/2}}{7} - \frac{2(\cos x)^{3/2}}{3} + c$$

6) $\int \frac{\sqrt{x}}{1+x}\, dx$ $\qquad x = u^2 \Rightarrow dx = 2u\, du$

$$= 2\int \frac{u^2}{1+u^2}\, du = 2\int \left(1 - \frac{1}{1+u^2}\right) du$$

$$= 2(u - \tan^{-1} u) + c = 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + c$$

7) $\int \sqrt{1+e^x}\, dx$ $\qquad 1+e^x = u^2 \Rightarrow e^x\, dx = 2u\, du$

$e^x = u^2 - 1 \Rightarrow dx = \frac{2u}{u^2-1} du$

$$= \int u \frac{2u}{u^2-1} du = 2\int \frac{u^2}{u^2-1} du$$

$$= 2\int \frac{u^2-1+1}{u^2-1} du = 2\int \left(\frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\right) du$$

$$= 2\int \left(1 - \frac{1}{1-u^2}\right) du$$

$$= 2(u - \tanh^{-1} u) + c = 2\left(u - \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)\right) + c$$

$$= 2\left(\sqrt{1+e^x} + \frac{1}{2}\ln\left(\frac{1+\sqrt{1+e^x}}{1-\sqrt{1+e^x}}\right)\right) + c$$

$$\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)}$$
$$= \frac{A}{1-u} + \frac{B}{1+u}$$
$$= \frac{A(1+u) + B(1-u)}{(1-u)(1+u)}$$
$\Rightarrow \quad 1 = A(1+u) + B(1-u)$

let $u = 1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

let $u = -1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

$$\int \frac{1}{1-u^2} dx = \int \left(\frac{\frac{1}{2}}{1-u} + \frac{\frac{1}{2}}{1+u}\right) dx$$
$$= -\frac{1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u) = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$

8) $\int \frac{dx}{(4+x)\sqrt{x}}$ $\qquad x = u^2 \Rightarrow dx = 2u\, du$

$$= \int \frac{2u\, du}{(4+u^2)u} = 2\int \frac{du}{(4+u^2)} = 2\cdot\frac{1}{2}\tanh^{-1}\frac{u}{2} + c$$

$$= \tanh^{-1}\frac{\sqrt{x}}{2} + c$$

9) $$\int \frac{dx}{x^{\frac{1}{2}}(1+x^{1/3})} \qquad x = u^6 \Rightarrow dx = 6\,u^5\,du$$

$$= \int \frac{6\,u^5\,du}{u^3(1+u^2)} = 6\int \frac{u^2\,du}{(1+u^2)} = 6\int \frac{1+u^2-1}{(1+u^2)}du$$

$$= 6\int \left(1 - \frac{1}{1+u^2}\right)du = 6(u + \tanh^{-1} u) + c$$

$$= 6\left(x^{1/6} + \tanh^{-1} x^{1/6}\right) + c$$

10) $$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} \qquad x = u^6 \Rightarrow dx = 6\,u^5\,du$$

$$= \int \frac{6u^5\,du}{u^3-u^2} = 6\int \frac{u^3\,du}{u-1}$$

$$= 6\int \left(u^2+u+1+\frac{1}{u-1}\right)du = 6\left(\frac{1}{3}u^3+\frac{1}{2}u^2+u+\ln|u-1|\right)+c$$

$$= 6\left(\frac{1}{3}\sqrt{x}+\frac{1}{2}\sqrt[3]{x}+\sqrt[6]{x}+\ln\left|\sqrt[6]{x}-1\right|\right)+c$$

$$= 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+c$$

11) $$\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}} \qquad x = u^{12} \Rightarrow dx = 12\,u^{11}\,du$$

$$= \int \frac{12\,u^{11}\,du}{u^4+u^3} = \int \frac{12\,u^{11}\,du}{u^3(u+1)} = 12\int \left(\frac{u^8}{u+1}\right)du$$

$$= 12\int \left(u^7-u^6+u^5-u^4+u^3-u^2+u-1+\frac{1}{u+1}\right)du$$

$$= 12\left(\frac{u^8}{8}-\frac{u^7}{7}+\frac{u^6}{6}-\frac{u^5}{5}+\frac{u^4}{4}-\frac{u^3}{3}+\frac{u^2}{2}-u+\ln|u+1|\right)+c$$

$$= 12\left(\frac{u^{2/3}}{8}-\frac{u^{7/12}}{7}+\frac{\sqrt{x}}{6}-\frac{u^{5/12}}{5}+\frac{\sqrt[3]{x}}{4}-\frac{\sqrt[4]{x}}{3}+\frac{\sqrt[6]{x}}{2}-\sqrt[12]{x}+\ln\left|\sqrt[12]{x}+1\right|\right)+c$$

12) $\int \frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$ $\qquad y = 1+x^2 \Rightarrow dy = 2x\,dx \Rightarrow dx = \frac{dy}{2}$

$$= \frac{1}{2}\int \frac{dy}{\sqrt{y+\sqrt{y^3}}} = \frac{1}{2}\int \frac{dy}{\sqrt{y+y\sqrt{y}}} = \frac{1}{2}\int \frac{dy}{\sqrt{y}\,\sqrt{1+\sqrt{y}}}$$

$$= \int \frac{1}{2\sqrt{y}}\left(1+\sqrt{y}\right)^{-\frac{1}{2}} dy = 2\left(1+\sqrt{y}\right)^{\frac{1}{2}} + c$$

$$\therefore \int \frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}} = 2\sqrt{1+\sqrt{1+x^2}} + c$$

13) $\int \frac{\sin^2 x \cos x}{2-\sin x}\,dx$ $\qquad$ $,let\ u = \sin x \Rightarrow du = \cos x\ dx$

$$\begin{array}{r|l} & u+2 \\ \hline u-2 & u^2 \\ & u^2-2u \\ \hline & 2u \\ & 2u-4 \\ \hline & 4 \end{array}$$

$$= \int \frac{u^2}{2-u}\,du = -\int \frac{u^2}{u-2}\,du = -\int\left(u+2-\frac{4}{u-2}\right)du$$

$$= \int\left(\frac{4}{u-2} - u - 2\right)du = 4\ln|u-2| - \tfrac{1}{2}u^2 - 2u + c$$

$$= 4\ln|\sin x - 2| - \tfrac{1}{2}\sin^2 x - 2\sin x + c$$

14) $\int \frac{\sin^{-1}(x/2)}{(4-x^2)^{1/2}}\,dx$

$$let\ u = \sin^{-1}\tfrac{x}{2} \Rightarrow du = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}dx = \frac{1}{\sqrt{1-\frac{x^2}{4}}}dx = \frac{1}{\sqrt{\frac{4-x^2}{4}}}dx = \frac{2}{\sqrt{4-x^2}}dx$$

$$\therefore I = \tfrac{1}{2}\int \sin^{-1}(x/2)\frac{2}{\sqrt{4-x^2}}\,dx = \tfrac{1}{2}\int u\,du = \tfrac{1}{2}\times\tfrac{u^2}{2} + c = \tfrac{1}{2}\left(\sin^{-1}\tfrac{x}{2}\right)^2 + c$$

التكامل بالتجزيئ :

لا توجد قاعدة خاصة لإجراء تكامل حاصل ضرب دالتين ، إلا إنه في بعض الحالات تكون الدالتين المضروبتين إحداهما سهلة الاشتقاق والأخرى سهلة التكامل عندها نطبق القاعدة التالية (التكامل بالتجزيئ)

$$\boxed{\int u\ dv = u\,v - \int v\ du}$$

ويمكن إثبات القاعدة من خلال مشتقة حاصل ضرب دالتين على النحو التالي :

إذا كانت $u\,,v$ دالتين فى x

$$\frac{d}{dx}(u\ \cdot v) = u\ \frac{dv}{dx} + v\ \frac{du}{dx}$$

$$\int \frac{d}{dx}(u\ \cdot v)\ dx = \int u\ \frac{dv}{dx}\ dx + \int v\ \frac{du}{dx}\ dx$$

$$(u\ \cdot v) = \int u\, dv + \int v\ \ du$$

$$\int u\, dv = u\ \cdot v - \int v\ \ du$$

أمثلة /

1) $\int (5x+1)(3x+1)^5\, dx$

$u = 5x + 1 \Rightarrow du = 5\, dx$

$dv = (3x+1)^5 dx$

$\Rightarrow v = \int (3x+1)^5 dx = \frac{(3x+1)^6}{18}$

$$= (5x+1)\frac{(3x+1)^6}{18} - \int \frac{5(3x+1)^6}{18}\ dx$$

$$= \frac{1}{18}(5x+1)(3x+1)^6 - \frac{5}{18}\frac{1}{3}\frac{(3x+1)^7}{7} + c$$

$$= \frac{1}{18}(5x+1)(3x+1)^6 - \frac{5}{54}\ \frac{(3x+1)^7}{7} + c$$

$$2)\int \frac{3x}{\sqrt[3]{2x+1}}dx = \int 3x\,(2x+1)^{-\frac{1}{3}}\,dx$$

$u = 3x \quad \Rightarrow \quad du = 3\,dx$

$dv = (2x+1)^{-\frac{1}{3}}dx \Rightarrow v = \frac{3}{4}(2x+1)^{\frac{2}{3}}$

$$= \frac{9}{4}x(2x+1)^{\frac{2}{3}} - \frac{9}{4}\int (2x+1)^{\frac{2}{3}}\,dx$$

$$= \frac{9}{4}x(2x+1)^{\frac{2}{3}} - \frac{9}{4}\,\frac{1}{2}\,\frac{3(2x+1)^{\frac{5}{3}}}{5} + c = \frac{9}{4}x(2x+1)^{\frac{2}{3}} - \frac{27}{40}\,(2x+1)^{\frac{5}{3}} + c$$

$$3)\int x\ e^{x}\ dx$$

$u = x \quad \Rightarrow \quad du = dx$

$dv = e^{x}\,dx \quad \Rightarrow \quad v = e^{x}$

$$= x\ e^{x} - \int\ e^{x}\ dx = x\ e^{x} - \ e^{x} + c$$

$$4)\int x^{2}\ e^{3x}\ dx$$

$u = x^{2} \quad \Rightarrow \quad du = 2x\,dx$

$dv = e^{3x}\,dx \quad \Rightarrow \quad v = \frac{1}{3}e^{3x}$

$$= \frac{1}{3}\ x^{2}\ e^{3x} - \frac{2}{3}\int x\,e^{3x}\ dx$$ نكامل بالتجزيئ مرة أخرى

$u = x \quad \Rightarrow \quad du = dx$

$dv = e^{3x}\,dx \quad \Rightarrow \quad v = \frac{1}{3}e^{3x}$

$$= \frac{1}{3}\ x^{2}\ e^{3x} - \frac{2}{3}\left[\frac{1}{3}x\ e^{3x} - \frac{1}{3}\int\ e^{3x}\ \ dx\right]$$

$$= \frac{1}{3}\ x^{2}\ e^{3x} - \frac{2}{3}\left[\frac{1}{3}x\ e^{3x} - \frac{1}{9}\ e^{3x}\right] + c$$

$$= \frac{1}{3}\ x^{2}\ e^{3x} - \frac{2}{9}x\ e^{3x} - \frac{2}{27}\ e^{3x} + c$$

$$5)\int x^{5}\ e^{x^{3}}\ dx = \int\ x^{3}\ x^{2}\ e^{x^{3}}\ dx$$

$u = x^{3} \quad \Rightarrow \quad du = 3x^{2}\,dx$

$dv = x^{2}e^{x^{3}}\,dx \quad \Rightarrow \quad v = \frac{1}{3}e^{x^{3}}$

$$= \frac{1}{3}\ x^{3}\ e^{x^{3}} - \int x^{2}\ e^{x^{3}}\ dx$$

$$= \frac{1}{3}\ x^{3}\ e^{x^{3}} - \frac{1}{3}\ \ e^{x^{3}} + c$$

6) $\int x\cos x \; dx$

$u = x \Rightarrow du = dx$

$dv = \cos x \; dx \Rightarrow v = \sin x$

$$= x\sin x - \int \sin x \; dx = x\sin x + \cos x + c$$

7) $\int x^2 \sin x \; dx$

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = \sin x\,dx \Rightarrow v = -\cos x$

$$= -x^2\cos x - (-2)\int x\cos x \; dx$$

$u = x \Rightarrow du = dx$

$dv = \cos x \; dx \Rightarrow v = \sin x$

$$= -x^2\cos x + 2[x\sin x + \cos x] + c$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

8) $\int x^2\cos(\pi x) \; dx$

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = \cos\pi x\,dx \Rightarrow v = \frac{1}{\pi}\sin\pi x$

$$= \frac{1}{\pi}x^2\sin\pi x - \frac{2}{\pi}\int x\sin\pi x \; dx$$

$u = x \Rightarrow du = dx$

$dv = \sin\pi x \; dx \Rightarrow v = -\frac{1}{\pi}\cos\pi x$

$$= \frac{1}{\pi}x^2\sin\pi x - \frac{2}{\pi}\left[-\frac{x}{\pi}\cos\pi x + \frac{1}{\pi}\int\cos\pi x \; dx\right]$$

$$= \frac{1}{\pi}x^2\sin\pi x + \frac{2}{\pi^2}x\cos\pi x - \frac{2}{\pi^3}\sin\pi x + x$$

9) $\int \ln x \; dx$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$dv = dx \Rightarrow v = x$

$$= x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - x + c$$

10) $\int x^3\ln x \; dx$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$dv = x^3dx \Rightarrow v = \frac{x^4}{4}$

$$= \frac{x^4}{4}\ln x - \int\frac{x^3}{4}dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + c$$

$$11)\ \int \frac{\ln(\ln x)}{x}\,dx$$

نستخدم التكامل بالتعويض أولاً

$$z = \ln x \quad \Rightarrow \quad dz = \frac{1}{x}\,dx$$

$$\Rightarrow \int \ln(\ln x)\ \frac{1}{x}\,dx = \int \ln(z)\quad dz$$

$$u = \ln z \quad \Rightarrow \quad du = \frac{1}{z}\,dz$$

$$dv = dz \quad \Rightarrow \quad v = z$$

$$= z\ln z - \int z\cdot\frac{1}{z}\,dz = z\ln z - z + c$$

$$\therefore \int \frac{\ln(\ln x)}{x}\,dx \qquad = \ln x\ln(\ln x) - \ln x + c$$

$$12)\ \int x^3(\ln x)^2\ dx$$

$$u = (\ln x)^2 \quad \Rightarrow \quad du = \frac{2}{x}\ln x\ \,dx$$

$$dv = x^3\,dx \quad \Rightarrow \quad v = \frac{x^4}{4}$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\int x^3\ \ln x\ dx$$

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\ dx$$

$$dv = x^3\,dx \quad \Rightarrow \quad v = \frac{x^4}{4}$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\left[\frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3 dx\right]$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4\ln x + \frac{1}{8}\int x^3 dx$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4 + c$$

$$u = \cos(\ln x) \Rightarrow \quad du = -\frac{1}{x}\sin(\ln x)\,dx$$

$$dv = dx \qquad \Rightarrow \quad v = x$$

$$13)\ \int \cos(\ln x)\ dx$$

$$u = \sin(\ln x) \Rightarrow \quad du = \frac{1}{x}\cos(\ln x)\,dx$$

$$dv = dx \qquad \Rightarrow \quad v = x$$

$$I\ = x\cos(\ln x) + \int \sin(\ln x)\,dx$$

$$I\ = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ dx$$

$$I\ = x\cos(\ln x) + x\sin(\ln x) - I$$

$$2I\ = x\cos(\ln x) + x\sin(\ln x)$$

$$I\ = \frac{1}{2}[x\cos(\ln x) + x\sin(\ln x)] + c$$

14) $\int x\, e^{\sqrt{x}}\, dx$

نستخدم التعويض

$x = z^2 \Rightarrow du = 2z\, dz$

$$= \int z^2 e^z \cdot 2z\, dz = 2\int z^3 e^z\, dz$$

$$= 2[z^3 e^z - 3z^2 e^z + 6z e^z - 6e^z] + c$$

$$= 2e^z[z^3 - 3z^2 + 6z - 6] + c$$

$$\therefore I = 2e^{\sqrt{x}}[x\sqrt{x} - 3x + 6\sqrt{x} - 6] + c$$

	نفاضل	نكامل
$+$	z^3	e^z
$-$	$3z^2$	e^z
$+$	$6z$	e^z
$-$	6	e^z
	0	e^z

15) $\int e^x \cos x\, dx$

$u = e^x \Rightarrow du = e^x$
$dv = \cos x\, dx \Rightarrow v = \sin x$

$$I = e^x \sin x - \int e^x \sin x\, dx$$

$u = e^x \Rightarrow du = e^x$
$dv = \sin x\, dx \Rightarrow v = -\cos x$

$$= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\, dx\right]$$

$$I = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$I = e^x \sin x + e^x \cos x - I$$

$$2I = e^x \sin x + e^x \cos x + c$$

$$I = \frac{1}{2}[e^x \sin x + e^x \cos x] + c = \frac{1}{2} e^x [\sin x + \cos x] + c =$$

*** $\int x e^x \cos x\, dx$

$U = x \qquad dV = e^x \cos x\, dx$

$dU = dx \qquad V = \frac{1}{2}e^x(\sin x + \cos x)$

$$= \frac{1}{2}xe^x(\sin + \cos x) - \frac{1}{2}\int e^x(\sin x + \cos x)\, dx$$

$$= \frac{1}{2}xe^x(\sin + \cos x)$$
$$- \frac{1}{4}e^x(\sin x - \cos x + \sin x + \cos x) + C$$

$$= \frac{1}{2}xe^x(\sin x + \cos x) - \frac{1}{2}e^x \sin x + C.$$

$$\boxed{\begin{aligned}\int e^x \cos x\, dx &= \tfrac{1}{2}e^x(\sin x + \cos x) + C,\\ \int e^x \sin x\, dx &= \tfrac{1}{2}e^x(\sin x - \cos x) + C.\end{aligned}}$$

16) $\int e^{2x} \sin 3x \ dx$

$u = e^{2x} \Rightarrow du = 2e^{x}$
$dv = \sin 3x \ dx \Rightarrow v = -\frac{1}{3}\cos 3x$

$$I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x \ dx$$

$u = e^{2x} \Rightarrow du = 2e^{x}$
$dv = \cos 3x \ dx \Rightarrow v = \frac{1}{3}\sin 3x$

$$I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left[\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x \ dx\right]$$

$$I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9} \ I$$

$$\frac{13}{9} \ I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x + c$$

$$I = \frac{9}{13}\left[-\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x\right] + c$$

$$I = \frac{1}{13}[-3e^{2x}\cos 3x + 2 \ e^{2x}\sin 3x] + c$$

17) $\int x \ \sec^2 x \ dx$

$u = x \Rightarrow du = dx$
$dv = \sec^2 x \ dx \Rightarrow v = \tan x$

$$= x\tan x - \int \tan x \ dx = x\tan x - \ln|\sec x| + c$$

18) $\int x \ \sin^2 x \ dx = \int x\left(\frac{1-\cos 2x}{2}\right) dx = \frac{1}{2}\int (x - x\cos 2x) \ dx$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \int x\cos 2x \ dx\right]$$

$u = x \Rightarrow du = dx$
$dv = \cos 2x \ dx \Rightarrow v = \frac{1}{2}\sin 2x$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \left(\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x \ dx\right)\right]$$

$$= \frac{x^2}{4} - \frac{1}{4}x\sin 2x + \frac{1}{4}\int \sin 2x \ dx = \frac{x^2}{4} - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$$

19) $\int \tan^2 x \sec x \, dx = \int \tan x \; (\tan x \sec x) dx$

$u = \tan x \Rightarrow du = \sec^2 x \; dx$
$dv = \tan x \;\; \sec x \;\; dx \Rightarrow v = \sec x$

$$I = \tan x \sec x - \int \sec^3 x \;\; dx$$

$$I = \tan x \sec x - \int (1 + \tan^2 x) \sec x \;\; dx$$

$$I = \tan x \sec x - \int \sec x \;\; dx - \int \tan^2 x \;\; \sec x \;\; dx$$

$$2\,I = \tan x \sec x - \ln|\sec x + \tan x| + c$$

$$I = \frac{1}{2} \tan x \sec x - \frac{1}{2} \ln|\sec x + \tan x| + c$$

20) $\int \sec^3 x \;\; dx = \int \sec^2 x \sec x \;\; dx$

$u = \sec x \Rightarrow du = \sec x \;\; \tan x \;\; dx$
$dv = \sec^2 x \;\; \sec x \;\; dx \Rightarrow v = \tan x$

$$I = \sec x \tan x - \int \tan x \;\; \sec x \tan x \;\; dx$$

$$I = \sec x \tan x - \int \tan^2 x \;\; \sec x \;\; dx$$

$$I = \sec x \tan x - \int (\sec^2 x - 1) \;\; \sec x \;\; dx$$

$$I = \sec x \tan x - \int \sec^3 x \;\; dx + \int \sec x \;\; dx$$

$$2\;I = \sec x \tan x + \int \sec x \;\; dx$$

$$I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

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21) $\int \tan^{-1} x \; dx$

$$u = \tan^{-1} x \Rightarrow du = \frac{1}{1+x^2} dx$$
$$dv = dx \quad \Rightarrow \quad v = x$$

$$= x \; \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$= x \; \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx = x \; \tan^{-1} x - \frac{1}{2} \ln|1+x^2| + c$$

22) $\int \sin^{-1} x \; dx$

$$u = \sin^{-1} x \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx$$
$$dv = dx \quad \Rightarrow \quad v = x$$

$$= x \; \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \; \sin^{-1} x + \frac{1}{2} \int -2x \, (1-x^2)^{-\frac{1}{2}} dx \; = x \; \sin^{-1} x + \sqrt{1-x^2} + c$$

23) $\int (\sin^{-1} x)^2 \; dx$

$$u = (\sin^{-1} x)^2 \Rightarrow du = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} dx$$
$$dv = dx \quad \Rightarrow \quad v = x$$

$$= x(\sin^{-1} x)^2 - \int \sin^{-1} x \; \frac{2x}{\sqrt{1-x^2}} dx$$

$$u = \sin^{-1} x \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx$$
$$dv = \frac{2x}{\sqrt{1-x^2}} dx \quad \Rightarrow \quad v = -2\sqrt{1-x^2}$$

$$= x(\sin^{-1} x)^2 - \left[-2 \sin^{-1} x \; \sqrt{1-x^2} - \int (-2) \, dx\right]$$

$$= x(\sin^{-1} x)^2 + 2 \sin^{-1} x \; \sqrt{1-x^2} + \int (-2) \, dx$$

$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} \; \sin^{-1} x \; - 2x + c$$

24) $\int x^2 \tan^{-1} x \, dx$

$$u = \tan^{-1} x \Rightarrow du = \frac{1}{1+x^2} dx$$
$$dv = x^2 \, dx \Rightarrow v = \frac{x^3}{3}$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x + \frac{x}{1+x^2} \, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \left[\frac{x^2}{2} + \frac{1}{2} \ln|1+x^2| \right] + c$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln(1+x^2) + c$$

25) $\int x \, \operatorname{sech}^{-1} x \, dx$

$$u = \operatorname{sech}^{-1} x \Rightarrow du = \frac{-1}{x\sqrt{1-x^2}} dx$$
$$dv = x \, dx \Rightarrow v = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2} \int \frac{-x}{\sqrt{1-x^2}} \, dx$$

$$= \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2} \int -x \, (1-x^2)^{-\frac{1}{2}} \, dx$$

$$= \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{4} \int -2x \, (1-x^2)^{-\frac{1}{2}} \, dx$$

$$= \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2} \sqrt{1-x^2} + c$$

26) $\int x \sinh x \, dx$

	نفاضل	تكامل
$+$	x	$\sinh x$
$-$	1	$\cosh x$
	0	$\sinh x$

$$= x \cosh x - \sinh x + c$$

27) $\int x\ \sin x\ \sin 2x\ dx = \int x\ \sin x\,(2\sin x\ \cos x)\ dx$

$= 2\int x\ \sin^2 x\ \cos x\ dx$

$u = x \Rightarrow du = dx$

$dv = \sin^2 x\ \cos x\ dx \ \Rightarrow v = \frac{1}{3}\sin^3 x$

$= 2\left[\frac{1}{3}\ x\ \sin^3 x - \int \frac{1}{3}\sin^3 x\ dx\right]$

$= \frac{2}{3}\ x\ \sin^3 x - \frac{2}{3}\int \sin^3 x\ dx = \frac{2}{3}\ x\ \sin^3 x - \frac{2}{3}\int \sin^2 x\ \sin x\ dx$

$= \frac{2}{3}\ x\ \sin^3 x - \frac{2}{3}\int (1-\cos^2 x)\ \sin x\ dx$

$= \frac{2}{3}\ x\ \sin^3 x - \frac{2}{3}\int (\sin x - \cos^2 x\ \sin x)\ dx$

$= \frac{2}{3}\ x\ \sin^3 x - \frac{2}{3}\left[-\cos x + \frac{1}{3}\cos^3 x\right] + c$

$= \frac{2}{3}\ x\ \sin^3 x + \frac{2}{3}\cos x - \frac{2}{9}\cos^3 x + c$

28) $\int \ln(a^2+x^2)\ dx$

$u = \ln(a^2+x^2) \Rightarrow du = \frac{2x}{a^2+x^2}dx$

$dv = dx \ \Rightarrow v = x$

$= x\ln(a^2+x^2) - \int \left(x\ \frac{2x}{a^2+x^2}\right)\ dx$

$= x\ln(a^2+x^2) - 2\int \frac{x^2}{a^2+x^2}\ dx = x\ln(a^2+x^2) - 2\int \frac{x^2+a^2-a^2}{a^2+x^2}\ dx$

$= x\ln(a^2+x^2) - 2\int \left(1 - \frac{a^2}{a^2+x^2}\right)dx$

$= x\ln(a^2+x^2) - 2\int dx + 2a^2\int \frac{1}{a^2+x^2}dx$

$= x\ln(a^2+x^2) - 2x + 2a^2 \times \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$= x\ln(a^2+x^2) - 2x + 2a\ \tan^{-1}\frac{x}{a} + c$

29) $\int x\ln\sqrt{x+2}\ dx$

$$= \int x\ln(x+2)^{\frac{1}{2}}\ dx = \frac{1}{2}\int x\ln(x+2)\ dx$$

$u = \ln(x+2) \Rightarrow du = \frac{1}{x+2}dx$

$dv = x\,dx \Rightarrow v = \frac{1}{2}x^2$

$$= \frac{1}{2}\left[\frac{1}{2}x^2\ln(x+2) - \int \frac{1}{2}x^2 \times \frac{1}{x+2}dx\right]$$

$$= \frac{1}{4}\left[x^2\ln(x+2) - \int \frac{x^2}{x+2}dx\right]$$

$$= \frac{1}{4}\left[x^2\ln(x+2) - \int\left(x-2+\frac{4}{x+2}\right)dx\right]$$

$$= \frac{1}{4}\left[x^2\ln(x+2) - \left(\frac{1}{2}x^2 - 2x + 4\ln(x+2)\right)\right]$$

$$= \frac{1}{4}\left[x^2\ln(x+2) - \frac{1}{2}x^2 + 2x - 4\ln(x+2)\right] + c$$

30) $\int e^{-x}\ \tan^{-1} e^{-x}\ dx$

$y = e^{-x} \Rightarrow dy = -\,e^{-x}\,dx$

$$\Rightarrow \int \tan^{-1} y\,(-dy) = -\int \tan^{-1} y\ dy$$

$u = \tan^{-1} y \Rightarrow du = \frac{1}{1+y^2}dy$

$dv = dy \Rightarrow v = y$

$$= -\left[y\ \tan^{-1} y - \int \frac{y}{1+y^2}\ dy\right]$$

$$= \frac{1}{2}\int \frac{2y}{1+y^2}\ dy - y\ \tan^{-1} y = \frac{1}{2}\ln(1+y^2) - y\ \tan^{-1} y + c$$

$$\therefore \int e^{-x}\ \tan^{-1} e^{-x}\ dx = \frac{1}{2}\ln(1+e^{-2x}) - e^{-x}\ \tan^{-1} e^{-x} + c$$

تكامل بعض الدوال المثلثية :

الحالة الاولى : $\int \sin^n x \, dx \quad , \quad \int \cos^n x \, dx \quad , \quad n =$ فردي

$$\int \sin^n x \, dx = \int \sin x \, \sin^{n-1} x \, dx \qquad , \sin^2 x = 1 - \cos^2 x$$

$$\int \cos^n x \, dx = \int \cos x \, \cos^{n-1} x \, dx \qquad , \cos^2 x = 1 - \sin^2 x$$

أمثلة :

$$1 - \int \sin^3 x \, dx = \int \sin x \, \sin^2 x \, dx = \int \sin x \, (1 - \cos^2 x) \, dx$$

$$= \int (\sin x - \cos^2 x \, \sin x) \, dx = -\cos x - \frac{\cos^3 x}{3} + c$$

$$2 - \int \cos^5 2x \, dx = \int \cos 2x \, \cos^4 2x \, dx = \int \cos 2x \, (\cos^2 2x)^2 \, dx$$

$$= \int \cos 2x \, (1 - \sin^2 2x)^2 \, dx = \int \cos 2x \, (1 - 4\sin^2 2x + \sin^4 2x) \, dx$$

$$= \int (\cos 2x - 4\sin^2 2x \, \cos 2x + \sin^4 2x \, \cos 2x) \, dx$$

$$= \frac{1}{2} \sin 2x - \frac{1}{2} \cdot 4 \cdot \frac{\sin^3 x}{3} + \frac{1}{2} \cdot \frac{\sin^5 x}{5} + c = \frac{1}{2} \sin 2x - 2 \cdot \frac{\sin^3 x}{3} + \frac{\sin^5 x}{10} + c$$

الحالة الثانية : $\int \sin^n x \, dx \quad , \quad \int \cos^n x \, dx \quad , \quad n =$ زوجي

أستخدم العلاقات : $\cos^2 x = \frac{1}{2}(1 + \cos 2x) \; , \; \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$3 - \int \cos^2 3x \, dx = \frac{1}{2} \int (1 + \cos 6x) \, dx = \frac{1}{2}\left(x + \frac{1}{6} \sin 6x\right) + c$$

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$$4-\int \sin^4 x\ dx = \int (\sin^2 x)^2\, dx = \frac{1}{4}\int (1-\cos 2x)^2\, dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\ dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right)\ dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)\ dx \quad = \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + c$$

الحالة الثالثة : $\int sin^n x\ cos^m x\ dx \quad , \quad n = \text{فردي}, m = \text{فردي}$

$$*If\ :\ n > m \Rightarrow I = \int sin^n x\ cos^m x\ dx = \int \sin x\ sin^{n-1} x\ cos^m x\ dx$$

$$*If\ :\ m > n \Rightarrow I = \int sin^n x\ cos^m x\ dx = \int sin^n x\ cos^{m-1} x \cos x\ dx$$

حيث: $sin^2 x = 1 - cos^2 x \quad , \quad cos^2 x = 1 - sin^2 x$

$$5-\int \sin^3 x\ \cos^3 x\ dx = \int \sin x \sin^2 x\ \cos^3 x\ dx = \int \sin x \sin^2 x\ \cos^3 x\ dx$$

$$= \int sin\, x\,(1 - cos^2 x)\ cos^3 x\ dx = \int (cos^3 x\, sin\, x - cos^5 x\, sin\, x)\ dx$$

$$6-\int sin^3 2x\ cos^5 2x\ dx = \int sin^3 2x\ cos\, 2x\ cos^4 2x\ dx$$

$$= \int sin^3 2x\ cos\, 2x\left(1 - sin^2 2x\right)^2\ dx$$

$$= \int sin^3 2x\ cos\, 2x\left(1 - 2\, sin^2 2x + sin^4 2x\right)\ dx$$

$$= \int \left(sin^3 2x\ cos\, 2x - 2\, sin^5 2x\ \ cos\, 2x + sin^7 2x\ cos\, 2x\right)\ dx$$

$$= \frac{1}{2}\frac{sin^4 2x}{4} - \frac{sin^6 2x}{6} + \frac{1}{2}\frac{sin^8 2x}{8} + c$$

$$= \frac{1}{8} sin^4 2x - \frac{1}{6} sin^6 2x + \frac{1}{16}\ sin^8 2x + c$$

حاول بنفسك

$$\int sin^3 2x\ cos^5 2x\ dx$$

$$= \int sin\, 2x\ sin^2 2x\ cos^5 2x\ dx$$

$$= \int sin\, 2x\ \ (1 - cos^2 2x\,)\, cos^5 2x\ dx$$

الحالة الرابعة : $\int sin^n x\ cos^m x\ \ dx \quad , \quad n =$ زوجي $, m =$ زوجي

أستخدم العلاقات : $cos^2 x = \frac{1}{2}(1 + cos\, 2x\,) \ , \quad sin^2 x = \frac{1}{2}(1 - cos\, 2x\,)$

$$7 - \int sin^2 x\ \ cos^2 x\ \ dx = \frac{1}{4}\int (2\sin x \cos x)^2 dx = \frac{1}{4}\int (\sin 2x)^2 dx = \frac{1}{4}\int sin^2 2x\ dx$$

$$= \frac{1}{4}\cdot\frac{1}{2}\int (1 - cos\, 4x\,)\ dx = \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + c$$

$$8 - \int sin^4 x\ \ cos^4 x\ \ dx = \frac{1}{16}\int (2\sin x \cos x)^4\ \ dx = \frac{1}{16}\int (\sin 2x)^4 dx$$

$$= \frac{1}{16}\int (sin^2\, 2x)^2 dx = \frac{1}{16}\int \left(\frac{1}{2}(1 - cos\, 4x\,)\right)^2 dx$$

$$= \frac{1}{64}\int (1 - 2\cos 4x + cos^2\, 4x)\ dx\ = \frac{1}{64}\int \left(1 - 2\,cos\, 4x + \frac{1}{2}(1 + cos\, 8x\,)\right)\ dx$$

$$= \frac{1}{64}\int \left(\frac{3}{2} - 2\cos 4x + \frac{1}{2}\cos 8x\right)\ dx\ \ = \frac{1}{64}\left(\frac{3}{2}x - \frac{1}{2}\sin 4x + \frac{1}{16}\sin 8x\right) + c$$

$$9 - \int sin^2 x\ \ cos^4 x\ \ dx = \int \frac{1}{2}(1 - cos\, 2x\,)\left(\frac{1}{2}(1 + cos\, 2x\,)\right)^2\ \ dx$$

$$= \frac{1}{8}\int (1 - cos\, 2x\,)\,(1 + 2\,cos\, 2x + cos^2\, 2x\,)\,dx$$

$$= \frac{1}{8}\int (1 + cos\, 2x - cos^2\, 2x - cos^3\, 2x\,)\,dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int cos^2\, 2x\ \ dx - \int cos^3\, 2x\ \ dx\right]$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int \frac{1}{2}(1 + cos\, 4x\,)\ \ dx - \int (1 - sin^2\, 2x)\,cos\, 2x\ \ dx\right]$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) - \int (cos\, 2x - sin^2\, 2x\,cos\, 2x\,)\ \ dx\right]$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) - \left(\frac{1}{2}\ \sin 2x - \frac{1}{2}\ \frac{sin^3\, 2x}{3}\right)\right] + c$$

$$= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\ \sin 4x + \frac{sin^3\, 2x}{6}\right] + c = \frac{1}{16}\left[x - \frac{1}{4}\ \sin 4x + \frac{1}{3}\ sin^3\, 2x\right] + c$$

الحالة الخامسة : $\int sec^n x \ dx \ , \quad \int csc^n x \ dx \ , \quad n =$ زوجي

$$\Rightarrow \int sec^n x \ dx = \int sec^{n-2} x \ sec^2 x \ dx \ , \quad sec^2 x = 1 + tan^2 x$$

$$, \int csc^n x \ dx = \int csc^{n-2} x \ csc^2 x \ dx \ , \quad csc^2 x = 1 + cot^2 x$$

$$10 - \int_0^{\frac{\pi}{4}} sec^6 x \ dx = \int_0^{\frac{\pi}{4}} sec^4 x \ sec^2 x \ dx = \int_0^{\frac{\pi}{4}} (1 + tan^2 x)^2 \ sec^2 x \ dx$$

$$= \int_0^{\frac{\pi}{4}} (1 + 2 \ tan^2 x + tan^4 x) \ sec^2 x \ dx$$

$$= \int_0^{\frac{\pi}{4}} (sec^2 x + 2 \ tan^2 x \, sec^2 x + tan^4 x \ sec^2 x) \ dx$$

$$= \left[tan\, x + \frac{2}{3} tan^3 \ x + \frac{1}{5} tan^5 x \ \right]_0^{\frac{\pi}{4}} = \left[1 + \frac{2}{3} + \frac{1}{5} \right] - 0 = \frac{28}{15}$$

$$11 - \int csc^4 2x \ dx = \int csc^2 2x \ csc^2 2x \ dx = \int csc^2 2x \ (cot^2 2x + 1) \ dx$$

$$= \int (cot^2 2x \ csc^2 2x + csc^2 2x) \ dx = -\frac{1}{6} cot^3 2x - \frac{1}{2} cot\, 2x + c$$

$$12 - \int x \ sec^4 x^2 \ dx = \int x \ sec^2 x^2 \ sec^2 x^2 \ dx$$

$$= \int x \ sec^2 x^2 \, (1 + tan^2 x^2) \ dx = \int (x \ sec^2 x^2 + x \ sec^2 x^2 \ tan^2 x^2) \ dx$$

$$= \frac{1}{2} \int (2x \ sec^2 x^2 + 2x \ sec^2 x^2 \ tan^2 x^2) \ dx$$

$$= \frac{1}{2} \left[tan\, x^2 + \frac{1}{3} tan^3 x^2) \right] + c$$

الحالة السادسة : $\int sec^n x \quad dx \quad , \quad \int csc^n x \quad dx \quad , \quad n =$ فردي

$$\int sec^n x \quad dx = \int \underbrace{sec\, x}_{u} \quad \underbrace{sec^{n-1} x \quad dx}_{dv}$$ (تكامل بالتجزيء)

$$\int csc^n x \quad dx = \int \overbrace{csc\, x} \quad \overbrace{csc^{n-1} x \quad dx}$$

13 − $\int sec^3 x \quad dx = \int sec\, x \quad sec^2 x \quad dx$

$u = \sec x \Rightarrow du = \sec x \tan x \, dx$

$, dv = \sec^2 x \ dx \Rightarrow v = \tan x$

$$\therefore \int sec^3 x \quad dx = sec\, x\, tan\, x - \int tan^2 x\, sec\, x\, dx$$

$$= sec\, x\, tan\, x - \int (sec^3 x - 1)\, sec\, x\, dx = sec\, x\, tan\, x - \int (sec^3 x - sec\, x) dx$$

$$\int sec^3 x \quad dx = sec\, x\, tan\, x - \int sec^3 x \ dx + \int sec\, x\, dx$$

$$2 \int sec^3 x \quad dx = sec\, x\, tan\, x + ln|\, sec\, x + tan\, x\,| + c_1$$

$$\therefore \int sec^3 x \quad dx = \frac{1}{2}\, sec\, x\, tan\, x + \frac{1}{2}\, ln|\, sec\, x + tan\, x\,| + c$$

14 − $\int csc^3 2x \quad dx = \int csc\, x \quad csc^2 2x \quad dx$

$u = csc\, 2x \Rightarrow du = -2 \csc 2x \ \cot 2x \, dx$

$dv = csc^2\, 2x \ dx \Rightarrow v = -\frac{1}{2} cot\, 2x$

$$= -\frac{1}{2}\, csc\, 2x\, cot\, 2x - \int csc\, 2x \quad cot^2 2x \ dx$$

$$= -\frac{1}{2}\, csc\, 2x\, cot\, 2x - \int csc\, 2x\, (csc^2 2x - 1) \ dx$$

$$= -\frac{1}{2}\, csc\, 2x\, cot\, 2x - \int csc^3 2x\, dx + \int csc\, 2x \quad dx$$

$$2 \int csc^3 2x \quad dx = -\frac{1}{2}\, csc\, 2x\, cot\, 2x + \frac{1}{2} ln|csc\, 2x - cot\, 2x\,| + c$$

$$\therefore \int csc^3 2x \quad dx = -\frac{1}{4}\, csc\, 2x\, cot\, 2x + \frac{1}{4} ln|csc\, 2x - cot\, 2x\,| + c$$

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الحالة السابعة ، $\int cot^n x \quad dx$ ، $\int tan^n x \quad dx$

$$\int tan^n x \quad dx = \int tan^{n-2} x \; tan^2 x \;\; dx \quad , \qquad tan^2 x = sec^2 x - 1$$

$$\int cot^n x \quad dx = \int cot^{n-2} x \;\; cot^2 x \; dx \quad , \qquad cot^2 x = csc^2 x - 1$$

$$15 - \int cot^3 2x \;\; dx = \int cot^2 2x \cot 2x \;\; dx = \int (csc^2 2x - 1) \cot 2x \;\; dx$$

$$= \int (cot\, 2x\, csc^2 2x - cot\, 2x) \;\; dx = -\frac{1}{2}\, cot^2 2x - \frac{1}{2} ln|sin\, 2x| + c$$

$$16 - \int x \; tan^4 x^2 \;\; dx = \int x \; tan^2 x^2 \;\; tan^2 x^2 \;\; dx = \int x \; tan^2 x^2 \;\; (sec^2 x^2 - 1) \;\; dx$$

$$= \int (x \; tan^2 x^2 \, sec^2 x^2 - x \; tan^2 x^2) \;\; dx$$

$$= \int x \; tan^2 x^2 \, sec^2 x^2 \;\; dx - \int x (sec^2 x^2 - 1) \;\; dx$$

$$= \frac{1}{2} \cdot \frac{1}{3} \; tan^3 x^2 - \frac{1}{2} \; tan \, x^2 + \frac{1}{2} \;\; x^2 + \; c$$

$$17 - \int tan^5 x \;\; dx = \int tan^3 x \;\; tan^2 x \;\; dx \; = \int tan^3 x \;\; (sec^2 x - 1) \;\; dx$$

$$= \int (tan^3 x \;\; sec^2 x - tan^3 x) \;\; dx$$

$$= \int tan^3 x \;\; sec^2 x \;\; dx - \int tan^3 x \, dx$$

$$= \frac{1}{4} \; tan^4 x - \int \tan x \, (sec^2 x - 1) \, dx$$

$$= \frac{1}{4} \; tan^4 x - \int (tan\, x \;\; sec^2 x - tan\, x) \, dx$$

$$= \frac{1}{4} \; tan^4 x - \frac{1}{2} \; tan^2 x + ln|\sec x| + c$$

الحالة الثامنة ، $\int cot^n x \; csc^m x \; dx$ ، $\int tan^n x \; sec^m x \; dx$

إذا كانت m = زوجي و n = فردي أو زوجي : $\int tan^n x \; sec^m x \; dx = \int tan^n x \; sec^2 x \; sec^{m-2} x \; dx$

إذا كانت m, n = فردي : $\int tan^n x \; sec^m x \; dx = \int tan^{n-1} x \; sec^{m-1} x \tan x \sec x \, dx$

وبالمثل $\int cot^n x \; csc^m x \; dx$

$$18 - \int sec^4 x \, tan^5 x \; dx = \int sec^2 x \; sec^2 x \; tan^5 x \; dx$$

$$= \int sec^2 x \,(1 + tan^2 x) tan^5 x \; dx$$

$$= \int (\, sec^2 x \; tan^5 x + \; sec^2 x \; tan^7 x) \; dx = \frac{1}{6} \, tan^6 x \; + \frac{1}{8} \; tan^8 x + \; c$$

$$19 - \int cot^3 x \, csc^5 x \; dx = \int cot^2 x \, csc^4 x \;\; \csc x \; \cot x \; dx$$

$$= \int (csc^2 x - 1) \; csc^4 x \;\; \csc x \; \cot x \; dx$$

$$= \int (csc^6 x \;\; \csc x \; \cot x - csc^4 x \;\; \csc x \; \cot x) \; dx$$

$$= -\frac{1}{7} \, csc^7 x + \frac{1}{5} \, csc^5 x \; + c$$

$20 - \int sec^4 x \;\; \sqrt{\tan x} \;\; dx$ للطالب

الحالة التاسعة : $\int \sqrt{1+\cos Ax}\ dx \ , \int \sqrt{1-\cos Ax}\ dx$

استخدم التعويض : $1+\cos Ax = 2\cos^2(\frac{A}{2}x)\ ,\ 1-\cos Ax = 2\sin^2\left(\frac{A}{2}x\right)$

$$21-\int \sqrt{1+\cos 4x}\ dx = \int \sqrt{2\cos^2(2x)}\ dx = \int \sqrt{2}\cos 2x\ dx$$

$$= \sqrt{2}\cdot\frac{1}{2}\ \sin 2x + c$$

$$22-\int \sqrt{2-2\cos 3x}\ dx = \sqrt{2}\int \sqrt{1-\cos 3x}\ dx = \sqrt{2}\int \sqrt{2\sin^2\left(\frac{3}{2}x\right)}\ dx$$

$$= 2\int \sin\left(\frac{3}{2}x\right)\ dx = 2\cdot\frac{-2}{3}\cos\left(\frac{3}{2}x\right) + c = -\frac{2}{3}\cos\left(\frac{3}{2}x\right) + c$$

$$23-\int \frac{1}{\sqrt{1+\cos 4x}}\, dx = \int \frac{1}{\sqrt{2\cos^2(2x)}}\, dx = \int \frac{1}{\sqrt{2}\cos 2x}\, dx$$

$$= \frac{1}{\sqrt{2}}\int \sec 2x\ dx = \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\ln|\sec 2x + \tan 2x\,| + c$$

طريقة الكسور الجزئية

إذا كانت $f(x) = \frac{q(x)}{r(x)}$ دالة قياسية أى أن $q(x), r(x)$ كثيرتي حدود، فعندما نكامل $f(x)$ وكانت درجة البسط أكبر من درجة المقام نقوم بعملية القسمة المطولة للحصول على $f(x) = g(x) + \frac{h(x)}{r(x)}$ حيث $h(x)$ دالة كثيرة حدود بدرجة أقل من درجة $r(x)$ ، ونحلل $r(x)$ إلى عوامل خطية أو عوامل تربيعية وهناك أربع حالات تعتمد على حالة معاملات المقام .

حالة I : إذا كان المقام عبارة عن حاصل ضرب عوامل خطية

لكل معامل خطي يوجد كسر جزئي وحيد على الشكل $\frac{A}{ax+b}$

$$\frac{q(x)}{(ax+b)(cx+d)\cdots(kx+l)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(cx+d)} + \cdots + \frac{A_3}{(kx+l)}$$

أمثلة /

$$1)\int \frac{dx}{x^2-4}$$

$$\because \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

يمكن حساب الثوابت A و B بواسطة طريقتين :

الطريقة العامة : بمساواة معاملات x التي لها نفس الأس .

$$\Rightarrow 1 = A(x+2) + B(x-2)$$

$$\Rightarrow 1 = Ax + 2A + Bx - 2B \Rightarrow 1 = x(A+B) + 2A - 2B$$

$$\Rightarrow \quad A + B = 0 \quad ① \quad , 2A - 2B = 1 \Rightarrow A - B = \frac{1}{2} \quad ②$$

وبحل المعادلتين : $A = \frac{1}{4} \; , B = -\frac{1}{4}$

الطريقة القصيرة : عوض في المعادلة عن 2 و -2 $x = 2$ و $x = -2$

$$x = 2 \Rightarrow \; 1 = 4A \; \Rightarrow A = \frac{1}{4} \qquad , \; x = -2 \Rightarrow \; 1 = -4B \; \Rightarrow B = -\frac{1}{4}$$

$$\therefore \frac{1}{x^2-4} = \frac{1/4}{x-2} + \frac{-1/4}{x+2}$$

$$\Rightarrow \int \frac{dx}{x^2-4} = \int \left(\frac{1/4}{x-2} + \frac{-1/4}{x+2}\right) dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + c$$

$$2) \int \frac{x^2+2x+6}{x^3-7x+14x-8}\,dx$$

$$\because \frac{x^2+2x+6}{x^3-7x+14x-8} = \frac{x^2+2x+6}{(x-2)(x-1)(x-4)} = \frac{A}{x-2}+\frac{B}{x-1}+\frac{C}{x-4}$$

$$= \frac{A(x-1)(x-4)+B(x-2)(x-4)+C(x-2)(x-1)}{(x-2)(x-1)(x-4)}$$

$$\Longrightarrow \quad x^2+2x+6 = A(x-1)(x-4)+B(x-2)(x-4)+C(x-2)(x-1)$$

$$put\ x=2 \Longrightarrow 14=-2A \quad \Longrightarrow A=-7$$

$$put\ x=1 \Longrightarrow 9=3B \quad \Longrightarrow B=3$$

$$put\ x=4 \Longrightarrow 30=6c \quad \Longrightarrow C=5$$

$$\therefore \int \frac{x^2+2x+6}{x^3-7x+13x-8}\,dx = \int \left(\frac{-7}{x-2}+\frac{3}{x-1}+\frac{5}{x-4}\right)dx$$

$$= -7\ln|x-2|+3\ln|x-1|+5\ln|x-4|+c = \ln\left|\frac{(x-1)^3(x-4)^5}{(x-2)^7}\right|+c$$

حالة II : العوامل الخطية المكررة

لكل معامل خطي $ax+b$ يحدث n من المرات في مقام كسر جزئي يكون هناك مجموع n من الكسور الجزئية .

$$\frac{q(x)}{(ax+b)^n} = \frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\cdots+\frac{A_n}{(ax+b)^n}$$

3) $\int \frac{2x^2-3x+3}{x^3-2x^2+x}\,dx$

$$\because \frac{2x^2-3x+3}{x^3-2x^2+x}=\frac{2x^2-3x+3}{x(x^2-2x+1)}=\frac{2x^2-3x+3}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$=\frac{A(x-1)^2+Bx(x-1)+C\,x}{x(x-1)^2}$$

$$\therefore\quad 2x^2-3x+3=A(x-1)^2+Bx(x-1)+C\,x$$

$put\ \ x=0 \implies \quad 3=A \implies A=3\quad ,\quad put\ \ x=1 \implies \quad 2=C \implies C=2$

$put\ \ x=2 \implies \quad 8-6+3=3+2B+2(2) \implies B=-1$

$$\therefore \int \frac{2x^2-3x+3}{x^3-2x^2+x}\,dx=\int\left(\frac{3}{x}-\frac{1}{x-1}+\frac{2}{(x-1)^2}\right)dx$$

$$=\int\left(\frac{3}{x}-\frac{1}{x-1}+2(x-1)^{-2}\right)dx=3\ln|x|-\ln|x-1|+2\,\frac{(x-1)^{-1}}{-1}+c$$

$$=3\ln|x|-\ln|x-1|-\frac{2}{x-1}+c$$

4) $\int \frac{x^3+1}{x(x-1)^3}\,dx$

$$\because \frac{x^3+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$$

$$=\frac{A(x-1)^3+Bx(x-1)^2+C\,x(x-1)+D\,x}{x(x-1)^3}$$

$$\implies x^3+1=A(x-1)^3+Bx(x-1)^2+C\,x(x-1)+D\,x$$

$put\ \ x=0 \implies \ 1=-A \implies A=-1,\ \ put\ \ x=1 \implies \ 2=D \implies D=2$

$put\ \ x=2 \implies \ 9=-1+2B+2C+4 \implies B+C=3 \qquad$ ①

$put\ \ x=-1 \implies 0=\ 8-4B+2C-2 \implies 2B-C=3 \qquad$ ②

بحل المعادلتين ① و ② : $\quad 3B=6 \implies B=2\quad ,\quad C=1$

$$\therefore \int \frac{x^3-1}{x(x-1)^3}\,dx=\int\left(\frac{-1}{x}+\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}\right)dx$$

$$=-\ln|x|+2\ln|x-1|-\frac{1}{x-1}-\frac{1}{(x-1)^2}+c$$

حالة III : العوامل التربيعية

لكل معامل تربيعي غير قابل للتحليل ax^2+bx+c يوجد كسر جزئي وحيد على شكل

$$\frac{Ax+b}{ax^2+bx+c}$$

$$\frac{q(x)}{(a_1x^2+b_1x+c_1)\cdots(a_1x^2+b_1x+c_1)}=\frac{A_1x+B_1}{(a_1x^2+b_1x+c_1)}+\cdots+\frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)}$$

5) $\displaystyle\int\frac{x^3+3}{(1+x)(1+x^2)}\,dx$

بفك الأقواس في المقام نلاحظ أن درجة البسط والمقام متساوية ، نقسم قسمة خوارزمية

$$=\int\frac{x^3+3}{x^3+x^2+x+1}\,dx$$

$$=\int\left(1+\frac{-x^2-x+2}{x^3+x^2+x+1}\right)dx$$

$$\frac{x^3+3}{x^3+x^2+x+1}$$
$$=\frac{x^3+x^2-x^2+x-x+3}{x^3+x^2+x+1}$$
$$=\frac{x^3+x^2+x+1-x^2-x+2}{x^3+x^2+x+1}$$
$$=\frac{(x^3+x^2+x+1)+(-x^2-x+2)}{x^3+x^2+x+1}$$
$$=1+\frac{-x^2-x+2}{x^3+x^2+x+1}$$

$$\frac{-x^2-x+2}{x^3+x^2+x+1}=\frac{-x^2-x+2}{(1+x)(1+x^2)}$$

$$=\frac{A}{(1+x)}+\frac{Bx+C}{(1+x^2)}$$

$$=\frac{A(1+x^2)+(Bx+C)(1+x)}{(1+x)(1+x^2)}$$

$$\Longrightarrow\quad -x^2-x+2=A(1+x^2)+(Bx+C)\,(1+x)$$

$put\ \ x=-1\ \Longrightarrow\ \ 2=2A\Longrightarrow A=1\quad ,put\ \ x=0\ \Longrightarrow\ 2=1+C\ \ \Longrightarrow C=1$

$put\ \ x=2\ \Longrightarrow\ -4=5+(2B+1)(3)\Longrightarrow 6B+3=-9\Longrightarrow B=-2$

$$\therefore\int\frac{x^3+3}{(1+x)(1+x^2)}\,dx=\int\left(1+\frac{1}{(1+x)}+\frac{-2x+1}{(1+x^2)}\right)dx$$

$$=\int\left(1+\frac{1}{(1+x)}-\frac{2x}{(1+x^2)}+\frac{1}{(1+x^2)}\right)dx$$

$$=x+\ln|1+x|-\ln|1+x^2|+\tan^{-1}(x)+c$$

6) $\int \frac{x+1}{x^4+10x^2+9}dx$

$$\frac{x+1}{(x^2+1)(x^2+9)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+9)}$$

$$= \frac{(Ax+B)(x^2+9)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+9)}$$

$$\Longrightarrow \ x+1 = (Ax+B)(x^2+9)+(Cx+D)(x^2+1)$$

$$x+1 = Ax^3+9Ax+Bx^2+9B+Cx^3+Cx+Dx^2+D$$

$$x+1 = (A+C)x^3+(B+D)x^2+(9A+C)x+(9B+D)$$

بمساواة المعاملات :

$A+C=0 \quad (1)$

$9A+C=1 \quad (2)$

$B+D=0 \quad (3)$

$9B+D=1 \quad (4)$

بحل المعادلتين (1) و (2) :

$$A=\frac{1}{8} \quad , \quad C=-\frac{1}{8}$$

بحل المعادلتين (3) و (4) :

$$B=\frac{1}{8} \quad , \quad D=-\frac{1}{8}$$

$$\therefore \int \frac{x+1}{x^4+10x^2+9}dx = \int \left(\frac{\frac{1}{8}x+\frac{1}{8}}{(x^2+1)} + \frac{-\frac{1}{8}x-\frac{1}{8}}{(x^2+9)} \right) dx$$

$$= \frac{1}{8}\int \left(\frac{x+1}{(x^2+1)} + \frac{-x-1}{(x^2+9)} \right) dx$$

$$= \frac{1}{8}\int \left(\frac{x}{(x^2+1)} + \frac{1}{(x^2+1)} - \frac{x}{(x^2+9)} - \frac{1}{(x^2+9)} \right) dx$$

$$= \frac{1}{8}\left[\frac{1}{2}\ln|x^2+1| + \tan^{-1}x - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\tan^{-1}\frac{x}{3} \right] + c$$

$$= \frac{1}{8}\left[\frac{1}{2}\ln\left|\frac{x^2+1}{x^2+9}\right| + \tan^{-1}x - \frac{1}{3}\tan^{-1}\frac{x}{3} \right] + c$$

7) $\int \frac{dx}{x^3-8}$

$$\frac{1}{x^3-8}=\frac{1}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$$

$$=\frac{A(x^2+2x+4)+(Bx+C)(x-2)}{(x-2)(x^2+2x+4)}$$

$$1=A(x^2+2x+4)+(Bx+C)(x-2)$$

$$put\ \ x=2\ \Rightarrow 1=12A\quad \Rightarrow A=\frac{1}{12}$$

$$put\ \ x=0\ \Rightarrow 1=\frac{1}{3}-2C\quad \Rightarrow C=-\frac{1}{3}$$

$$put\ \ x=1\ \Rightarrow 1=\frac{7}{12}+\left(B-\frac{1}{3}\right)(-1)\Rightarrow \frac{5}{12}=\frac{1}{3}-B\Rightarrow B=-\frac{1}{12}$$

$$\therefore \int\frac{dx}{x^3+8}=\int\left(\frac{\frac{1}{12}}{x-2}+\frac{-\frac{1}{12}x-\frac{1}{3}}{x^2+2x+4}\right)dx=$$

$$=\frac{1}{12}\ln|x-2|-\frac{1}{12}\int\left(\frac{x+4}{x^2+2x+4}\right)dx=\frac{1}{12}\ln|x-2|-\frac{1}{24}\int\left(\frac{2x+8}{x^2+2x+4}\right)dx$$

$$=\frac{1}{12}\ln|x-2|-\frac{1}{24}\int\left(\frac{2x+2+6}{x^2+2x+4}\right)dx$$

$$=\frac{1}{12}\ln|x-2|-\frac{1}{24}\int\left(\frac{2x+2}{x^2+2x+4}+\frac{6}{x^2+2x+4}\right)dx$$

$$=\frac{1}{12}\ln|x-2|-\frac{1}{24}\ln|x^2+2x+4|-\frac{1}{24}\int\left(\frac{6}{x^2+2x+1+3}\right)dx$$

$$=\frac{1}{12}\ln|x-2|-\frac{1}{24}\ln|x^2+2x+4|-\frac{1}{4}\int\frac{dx}{(x+1)^2+3}$$

$$=\frac{1}{12}\ln|x-2|-\frac{1}{24}\ln|x^2+2x+4|-\frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c$$

حالة IV : عوامل تربيعية مكررة

لكل معامل تربيعي غير قابل للتحليل $x^2 + bx + c$ يحدث n من المرات في مقام كسر جزئي صحيح يكون هناك مجموع عدد n من الكسور الجزئية

$$\frac{q(x)}{(ax^2+bx+c)^n} = \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

8) $\int \frac{x^3-2x}{(x^2+1)^2}\, dx$

$$\frac{x^3-2x}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} = \frac{(Ax+B)(x^2+1)+(Cx+D)}{(x^2+1)}$$

$$x^3-2x = (Ax+B)(x^2+1)+(Cx+D)$$

$$x^3-2x = Ax^3+Ax+Bx^2+B+Cx+D$$

$$x^3-2x = Ax^3+Bx^2+(A+C)x+B+D$$

$$\Longrightarrow A=1\ \ ,B=0\,,A+C=-2\ \ \Longrightarrow C=-2-1=-3$$

$$B+D=0 \Longrightarrow D=0$$

$$\therefore \int \frac{x^3-2x}{(x^2+1)^2}\, dx = \int \left(\frac{x}{(x^2+1)} + \frac{-3x}{(x^2+1)^2}\right) dx$$

$$= \int \left(\frac{x}{x^2+1}\right) dx + \int \left(\frac{3x}{(x^2+1)^2}\right) dx$$

$$= \frac{1}{2}\int \left(\frac{2x}{x^2+1}\right) dx - \frac{3}{2}\int 2x\,(x^2+1)^{-2}\ dx$$

$$= \frac{1}{2}\ln|x^2+1| - \frac{3}{2}\frac{(x^2+1)^{-1}}{-1} + c$$

$$= \frac{1}{2}\ln|x^2+1| + \frac{3}{2(x^2+1)} + c$$

9) $\int \frac{dx}{x^5 - x^2}$

$$\frac{1}{x^2(x^3-1)} = \frac{1}{x^2(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{Dx+E}{(x^2+x+1)}$$

$$= \frac{Ax(x-1)(x^2+x+1) + B(x-1)(x^2+x+1) + Cx^2(x^2+x+1) + (Dx+E)x^2(x-1)}{x^2(x-1)(x^2+x+1)}$$

$$= \frac{Ax(x^3-1) + B(x^3-1) + C(x^4+x^3+x^2) + (Dx+E)(x^3-x^2)}{x^2(x-1)(x^2+x+1)}$$

$$1 = Ax(x^3-1) + B(x^3-1) + C(x^4+x^3+x^2) + (Dx+E)(x^3-x^2)$$

$$1 = Ax^3 - Ax + Bx^3 - B + Cx^4 + Cx^3 + Cx^2 + Dx^4 - Dx^3 + Ex^3 - Ex^2$$

$$1 = (C+D)x^4 + (A+B+C-D+E)x^3 + (C-E)x^2 - Ax - B$$

$\because\ -B = 1 \Rightarrow B = -1\ , -A = 0 \Rightarrow A = 0$

$\because\ C + D = 0 \Rightarrow D = -C$ ① $\quad , C - E = 0 \Rightarrow E = C$ ②

$A + B + C - D + E = 0 \Rightarrow -1 + C - D + E = 0$ ③

بحل المعادلات من ① ، ② بالتعويض في ③ : $-1 + C + C + C = 0 \Rightarrow C = \frac{1}{3}$

بالتعويض في المعادلة ① و② : $E = \frac{1}{3}\ ,\quad D = -\frac{1}{3}$

$$\int \frac{dx}{x^5 - x^2} = \int \left(\frac{0}{x} + \frac{-1}{x^2} + \frac{\frac{1}{3}}{(x-1)} + \frac{-\frac{1}{3}x + \frac{1}{3}}{(x^2+x+1)} \right) dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \left(\frac{x-1}{(x^2+x+1)} \right) dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \left(\frac{2x-2}{(x^2+x+1)} \right) dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \left(\frac{2x+1-3}{(x^2+x+1)} \right) dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\left(\int \left(\frac{2x+1}{(x^2+x+1)} \right) dx - 3\int \left(\frac{1}{(x^2+x+1)} \right) dx \right)$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\int \frac{1}{(x+1/2)^2 + 3/4}\, dx$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2(x+1/2)}{\sqrt{3}} \right) + c$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}} \right) + c$$

10) $\int \frac{y^{\frac{2}{3}}\ dy}{y+1}$ $\qquad y = u^3 \Rightarrow \quad dy = 3u^2 du$

$$= \int \frac{u^2\ 3u^2 du}{u^3+1} = 3\int \frac{u^4\ du}{u^3+1} = \int \left(u - \frac{u}{u^3+1}\right) du$$

$$\frac{u}{u^3+1} = \frac{u}{(u+1)(u^2-u+1)} = \frac{A}{(u+1)} + \frac{Bu+C}{(u^2-u+1)}$$

$$= \frac{A(u^2-u+1)+(Bu+C)(u+1)}{(u+1)(u^2-u+1)}$$

$$u = A(u^2-u+1)+(Bu+C)(u+1)$$

$$\Rightarrow u = Au^2 - Au + A + Bu^2 + Cu + Bu + C$$

$$\Rightarrow u = (A+B)u^2 + (B+C-A)u + A + C$$

$A + C = 0 \Rightarrow A = -C \ldots..(1)$
$B + C - A = 1 \ldots\ldots (2)$
$A + B = 0 \Rightarrow B = -A \ldots..(3)$
$\xrightarrow{\text{بالتعويض في}(2)} C - 2A = 1 \ldots..(4)$
$\xrightarrow{\text{من }(1)\text{ و}(4)} C = \frac{1}{3} \Rightarrow A = -\frac{1}{3}\ , B = \frac{1}{3}$

$$\int \left(u - \frac{u}{u^3+1}\right) du = \int \left(u - \left(\frac{-\frac{1}{3}}{(u+1)} + \frac{\frac{1}{3}u+\frac{1}{3}}{u^2-u+1}\right)\right) du$$

$$\int \left(u - \frac{u}{u^3+1}\right) du = \int \left(u + \frac{\frac{1}{3}}{(u+1)} - \frac{\frac{1}{3}u+\frac{1}{3}}{u^2-u+1}\right) du$$

$$= \frac{1}{2}u^2 + \frac{1}{3}\ln|u+1| - \frac{1}{3}\int \left(\frac{u+1}{u^2-u+1}\right) du = \frac{1}{2}u^2 + \frac{1}{3}\ln|u+1| - \frac{1}{3}\frac{1}{2}\int \left(\frac{2u+2}{u^2-u+1}\right) du$$

$$= \frac{1}{2}u^2 + \frac{1}{3}\ln|u+1| - \frac{1}{6}\int \left(\frac{2u-1+3}{u^2-u+1}\right) du$$

$$= \frac{1}{2}u^2 + \frac{1}{3}\ln|u+1| - \frac{1}{6}\int \frac{2u-1}{u^2-u+1} du - \frac{1}{6}\int \frac{3}{u^2-u+1} du$$

$$= \frac{1}{2}u^2 + \frac{1}{3}\ln|u+1| - \frac{1}{6}\int \frac{2u-1}{u^2-u+1} du - \frac{1}{2}\int \frac{1}{\left(u-\frac{1}{2}\right)^2 + \frac{3}{4}} du$$

$$= \frac{1}{2}u^2 + \frac{1}{3}\ln|u+1| - \frac{1}{6}\ln|u^2-u+1| - \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\frac{4\left(u-\frac{1}{2}\right)}{\sqrt{3}} + c$$

$$= \frac{1}{2}u^2 + \frac{1}{3}\ln|u+1| - \frac{1}{6}\ln|u^2-u+1| - \frac{1}{\sqrt{3}}\tan^{-1}\frac{(4u-2)}{\sqrt{3}} + c$$

$$\because\ y = u^3 \ \Rightarrow u = \sqrt[3]{y}$$

$$\therefore \int \frac{y^{\frac{2}{3}}\ dy}{y+1} = \frac{1}{2}\sqrt[3]{y^2} + \frac{1}{3}\ln\left|\sqrt[3]{y}+1\right| - \frac{1}{6}\ln\left|\sqrt[3]{y^2} - \sqrt[3]{y} + 1\right| - \frac{1}{\sqrt{3}}\tan^{-1}\frac{(4\sqrt[3]{y}-2)}{\sqrt{3}} + c$$

11) $\int \frac{e^{2x}\,dx}{e^{2x}+3e^x+2}$ $\qquad y = e^x \Rightarrow dy = e^x dx \Rightarrow dy = y\,dx$

$$= \int \frac{y^2\,\frac{dy}{y}}{y^2+3y+2} = \int \frac{y\,dy}{y^2+3y+2}$$

$$\frac{y}{y^2+3y+2} = \frac{y}{(y+1)(y+2)} = \frac{A}{(y+1)} + \frac{B}{(y+2)} = \frac{A(y+2)+B(y+1)}{(y+1)(y+2)}$$

$$\Rightarrow y = A(y+2) + B(y+1)$$

$put\ y=-1 \Rightarrow -1 = A \Rightarrow A = -1$, $put\ y=-2 \Rightarrow -2 = -B \Rightarrow B = 2$

$$\therefore \int \frac{y\,dy}{y^2+3y+2} = \int \left(\frac{-1}{(y+1)} + \frac{2}{(y+2)}\right) dy = -\ln|y+1| + 2\ln|y+2| + c$$

$$= \ln\left|\frac{(y+2)^2}{y+1}\right| + c$$

$$\because y = e^x \Rightarrow \int \frac{e^{2x}\,dx}{e^{2x}+3e^x+2} = \ln\left[\frac{(e^x+2)^2}{e^x+1}\right] + c$$

12) $\int \frac{\sin x\,dx}{\cos^2 x - 5\cos x + 4}$ $\qquad y = \cos x \Rightarrow dy = -\sin x\,dx$

$$= \int \frac{-dy}{y^2-5y+4}$$

$$\frac{-1}{y^2-5y+4} = \frac{-1}{(y-1)(y-4)} = \frac{A}{(y-1)} + \frac{B}{(y-4)} = \frac{A(y-4)+B(y-1)}{(y-1)(y-4)}$$

$$\Rightarrow -1 = A(y-4) + B(y-1)$$

$put\ y=1 \Rightarrow -1 = -3A \Rightarrow A = \frac{1}{3}$, $put\ y=4 \Rightarrow -1 = 3B \Rightarrow B = -\frac{1}{3}$

$$\therefore \int \frac{dy}{y^2-5y+4} = \int \left(\frac{1/3}{y-1} - \frac{1/3}{y-4}\right) dy = \frac{1}{3}\ln|y-1| - \frac{1}{3}\ln|y-4| + c$$

$$= \frac{1}{3}\ln\left|\frac{y-1}{y-4}\right| + c$$

$$\because y = \cos x \Rightarrow \int \frac{\sin x\,dx}{\cos^2 x - 5\cos x + 4} = \frac{1}{3}\ln\left|\frac{\cos x - 1}{\cos x - 4}\right| + c$$

13) $\int \frac{\cos x \ dx}{\sin^2 x + \sin x - 6}$

$$= \int \frac{dy}{y^2 + y - 6} \qquad\qquad y = \sin x \ \Rightarrow dy = \cos x \ dx$$

$$\frac{1}{y^2 + y - 6} = \frac{1}{(y+3)(y-2)} = \frac{A}{(y+3)} + \frac{B}{(y-2)} = \frac{A(y-2) + B(y+3)}{(y+3)(y-2)}$$

$$\Rightarrow \ 1 = A(y-2) + B(y+3)$$

$put \ \ y = -3 \ \Rightarrow 1 = -5A \quad \Rightarrow A = -1/5 \ , put \ \ y = 2 \ \Rightarrow \ 1 = 5\,B \Rightarrow B = \ 1/5$

$$\therefore \int \frac{dy}{y^2 + y - 6} = \int \left(\frac{-1/5}{(y+3)} + \frac{1/5}{(y-2)} \right) dy = -\frac{1}{5} \ln|y+3| + \frac{1}{5} \ln|y-2| + c$$

$$= \frac{1}{5} \ln \left| \frac{y-2}{y+3} \right| + c$$

$$\because \ \ y = \sin x \ \Rightarrow \int \frac{\cos x \ dx}{\sin^2 x + \sin x - 6} = \frac{1}{5} \ln \left| \frac{\sin x - 2}{\sin x + 3} \right| + c$$

14) $\int \frac{\sin x \ dx}{\cos^2 x + \cos x - 2}$

$$\int \frac{-dy}{y^2 + y - 2} \qquad\qquad y = \cos x \ \Rightarrow dy = -\sin x \ dx$$

$$\frac{-1}{y^2 + y - 2} = \frac{A}{(y+2)} + \frac{B}{(y-1)} = \frac{A(y-1) + B(y+2)}{(y+2)(y-1)}$$

$$\Rightarrow \ -1 = A(y-1) + B(y+2)$$

$put \ \ y = 1 \Rightarrow -1 = 3B \quad \Rightarrow B = -1/3, put \ \ y = -2 \ \Rightarrow \ -1 = -3\,A \Rightarrow A = \ 1/3$

$$\therefore \int \frac{-dy}{y^2 + y - 2} = \int \left(\frac{1/3}{(y+2)} + \frac{-1/3}{(y-1)} \right) dy = \frac{1}{3} \ln|y+2| - \frac{1}{3} \ln|y-1| + c$$

$$\because \ y = \cos x \quad \Rightarrow \int \frac{\sin x \ dx}{\cos^2 x + \cos x - 2} = \frac{1}{3} \ln \left| \frac{\cos x + 2}{\cos x - 1} \right| + c$$

$$15)\int \frac{dx}{\sec^2 x+\tan^2 x}=\int \frac{dx}{1+\tan^2 x+\tan^2 x}$$

$$y=\tan x \quad \Rightarrow dy=\sec^2 x\ dx$$

$$=\int \frac{dx}{1+2\tan^2 x} \qquad \Rightarrow dy=(1+\tan^2 x)\ dx$$

$$=\int \frac{dy}{(1+y^2)(1+2y^2)} \qquad \Rightarrow dy=(1+y^2)\ dx \Rightarrow dx=\frac{dy}{(1+y^2)}$$

$$\frac{1}{(1+y^2)(1+2y^2)}=\frac{Ay+B}{(1+y^2)}+\frac{Cy+D}{(1+2y^2)}$$

$$=\frac{(Ay+B)(1+2y^2)+(Cy+D)(1+y^2)}{(1+y^2)(1+2y^2))}$$

$$\Rightarrow 1=(Ay+B)(1+2y^2)+(Cy+D)(1+y^2)$$

$$1=Ay+B+2A\,y^3+2B\,y^2+Cy+D+C\,y^3+Dy^2$$

$$1=(2A+C)\,y^3+(2B+D)\,y^2+(A+C)y+(B+D)$$

$$\Rightarrow \quad 2A+C=0 \ \ldots\ldots(1) \ , \ A+C=0 \ldots\ldots(2) \xRightarrow{\text{بحل المعادلتين}} A=0\,,C=0$$

$$,2B+D=0 \ldots\ldots(3) \ , \ B+D=1 \ldots\ldots(4) \xRightarrow{\text{بحل المعادلتين}} B=-1,D=2$$

$$\therefore \int \frac{dy}{(1+y^2)(1+2y^2)}=\int\left(\frac{-1}{1+y^2}+\frac{2}{1+2y^2}\right)dy$$

$$=\int\left(\frac{-1}{1+y^2}+\sqrt{2}\frac{\sqrt{2}}{1+\left(\sqrt{2}y\right)^2}\right)dy$$

$$=-\tan^{-1} y+\sqrt{2}\ \tan^{-1}\left(\sqrt{2}y\right)+c$$

$$\because y=\tan x \ \Rightarrow \int \frac{dx}{\sec^2 x+\tan^2 x}=-\tan^{-1}(\tan x)+\sqrt{2}\ \tan^{-1}\left(\sqrt{2}\ \tan x\right)+c$$

$$=-x+\sqrt{2}\ \tan^{-1}\left(\sqrt{2}\ \tan x\right)+c$$

$$16)\int \frac{\tan^{-1}x}{x^2}\,dx$$

$$u=\tan^{-1}x \Rightarrow du=\frac{1}{1+x^2}dx$$
$$dv=\frac{1}{x^2}dx \;\Rightarrow\; v=-\frac{1}{x}$$

$$=\frac{1}{x}\tan^{-1}x-\int -\frac{1}{x}\times\frac{1}{1+x^2}dx$$

$$=\frac{1}{x}\tan^{-1}x+\int\frac{1}{x\,(1+x^2)}dx$$

$$\frac{1}{x\,(1+x^2)}=\frac{A}{x}+\frac{Bx+c}{1+x^2}=\frac{A(1+x^2)+x(Bx+C)}{x\,(1+x^2)}$$

$$\Rightarrow 1=A(1+x^2)+x(Bx+C)$$

$$\Rightarrow 1=A+Ax^2+Bx^2+Cx \Rightarrow 1=(A+B)x^2+Cx+A$$

$$\Rightarrow A=1\,,C=0\,,A+B=0 \Rightarrow B=-A=-1$$

$$\int\frac{\tan^{-1}x}{x^2}\,dx=\frac{1}{x}\tan^{-1}x+\int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx$$

$$=\frac{1}{x}\tan^{-1}x+\ln x-\frac{1}{2}\ln(1+x^2)+c$$

17) $\int \frac{\cos x \; dx}{\sin^2 x\,(3+2\sin x)}$

$$I = \int \frac{dy}{y^2(3+2y)} \qquad\qquad y=\sin x \;\Rightarrow dy=\cos x\,dx$$

$$\frac{1}{y^2(3+2y)}=\frac{A}{y}+\frac{B}{y^2}+\frac{C}{3+2y}=\frac{Ay(3+2y)+B(3+2y)+Cy^2}{y^2(3+2y)}$$

$$\Rightarrow\; 1=Ay\,(3+2y)+B(3+2y)+Cy^2$$

$$\Rightarrow\; 1=3Ay+2Ay^2+3B+2By+Cy^2$$

$$\Rightarrow\; 1=(2A+C)y^2+(3A+2B)y+3B$$

$$\Rightarrow 1=3B \quad ,3A+2B=0 \;\;,2A+C=0$$

$$\Rightarrow B=\frac{1}{3} \quad ,3A=-2B \Rightarrow 3A=-\frac{2}{3} \Rightarrow A=-\frac{2}{9}\,,C=-2A \Rightarrow C=\frac{4}{9}$$

$$\therefore \int \frac{dy}{y^2(3+2y)}=\int\left(\frac{-\frac{2}{9}}{y}+\frac{\frac{1}{3}}{y^2}+\frac{\frac{4}{9}}{3+2y}\right)dy$$

$$=-\frac{2}{9}\ln|y|-\frac{1}{3y}+\frac{4}{9}\times\frac{1}{2}\ln|3+2y|+c=\frac{2}{9}(\ln|3+2y|-\ln|y|)-\frac{1}{3y}+c$$

$$=\frac{2}{9}\ln\left|\frac{3+2y}{y}\right|-\frac{1}{3y}+c$$

$$\therefore \int \frac{\cos x\; dx}{\sin^2 x\,(3+2\sin x)}=\frac{2}{9}\ln\left|\frac{3+2\sin x}{\sin x}\right|-\frac{1}{3\sin x}+c$$

التعويضات المثلثية

اذا احتوت التكاملات على جذور أومقلوبات إحدى الصور a^2-x^2 أو a^2+x^2 أو x^2-a^2 نستخدم التعويضات :

$a^2-x^2 \xrightarrow{\text{التعويض}} x = a\sin\theta \quad , \quad 1-\sin^2\theta = \cos^2\theta$

$a^2+x^2 \xrightarrow{\text{التعويض}} x = a\tan\theta \quad , \quad 1+\tan^2\theta = \sec^2\theta$

$x^2-a^2 \xrightarrow{\text{التعويض}} x = a\sec\theta \quad , \quad \sec^2\theta - 1 = \tan^2\theta$

أمثلة /

1) $\int \sqrt{4-x^2}\, dx$ $\qquad$ $let\ \ x = 2\sin\theta \Rightarrow dx = 2\cos\theta\ d\theta$

$$= \int \sqrt{4-(2\sin\theta)^2}\ \ 2\cos\theta\, d\theta = \int \sqrt{4-4\sin^2\theta}\ \ 2\cos\theta\, d\theta$$

$$= \int 2\sqrt{1-\sin^2\theta}\ \ 2\cos\theta\, d\theta = 4\int \sqrt{1-\sin^2\theta}\ \ \cos\theta\, d\theta$$

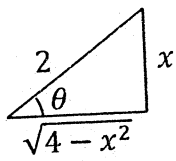
$$= 4\int \cos^2\theta\, d\theta = 4\int \frac{1+\cos 2\theta}{2}\, d\theta = 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + c$$

$$= 2\theta + \sin 2\theta + c = 2\theta + 2\sin\theta\cos\theta + c$$

$$\because x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$$

$$\therefore \int \sqrt{4-x^2}\, dx = 2\sin^{-1}\frac{x}{2} + 2\cdot\frac{x}{2}\ \frac{2}{\sqrt{4-x^2}} + c$$

$\cos\theta =$



$$= 2\sin^{-1}\frac{x}{2} + \frac{2x}{\sqrt{4-x^2}} + c$$

2) $\int \sqrt{9+x^2}\,dx$ $\qquad$ *let* $x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\, d\theta$

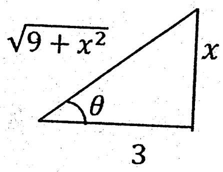
$$= \int \sqrt{9+9\tan^2\theta}\; 3\sec^2\theta\, d\theta = 9\int \sqrt{1+\tan^2\theta}\; \sec^2\theta\, d\theta$$

$$= 9\underbrace{\int \sec^3\theta\, d\theta}_{\text{تكامل بالتجزيئ}} = 9\cdot\frac{1}{2}[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|] + c$$

$\because x = 3\tan\theta \Rightarrow \tan\theta = \frac{x}{3}$

$$\therefore \int \sqrt{9+x^2}\,dx = \frac{1}{2}\left[\frac{\sqrt{9+x^2}}{3}\cdot\frac{x}{3} + \ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|\right] + c$$

$\sec\theta = \frac{\sqrt{9+x^2}}{3}$



$$= \frac{1}{2}\left[\frac{x\sqrt{9+x^2}}{9} + \ln\left|\frac{\sqrt{9+x^2}+x}{3}\right|\right] + c$$

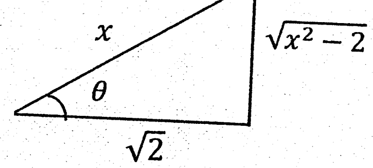
3) $\int \sqrt{x^2-2}\,dx$ $\qquad$ *let* $x = \sqrt{2}\sec\theta \Rightarrow dx = \sqrt{2}\sec\theta\tan\theta\, d\theta$

$$= \int \sqrt{2\sec^2\theta - 2}\;\sqrt{2}\,\sec\theta\tan\theta\, d\theta = 2\int \sqrt{\sec^2\theta - 1}\;\sec\theta\tan\theta\, d\theta$$

$$= 2\int \sec\theta\tan^2\theta\, d\theta = 2\cdot\frac{1}{2}[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|] + c$$

$\because\ x = \sqrt{2}\sec\theta \Rightarrow \sec\theta = \frac{x}{\sqrt{2}}$

$\tan\theta = \frac{\sqrt{x^2-2}}{\sqrt{2}}$



$$\therefore \int \sqrt{x^2-2}\,dx = \left[\frac{x}{\sqrt{2}}\cdot\frac{\sqrt{x^2-2}}{\sqrt{2}} - \ln\left|\frac{x}{\sqrt{2}}+\frac{\sqrt{x^2-2}}{\sqrt{2}}\right|\right] + c$$

$$= \left[\frac{x\sqrt{x^2-2}}{2} - \ln\left|\frac{x+\sqrt{x^2-2}}{\sqrt{2}}\right|\right] + c$$

4) $\int x^2\sqrt{1-x^2}\ \ dx$ $\qquad$ *let* $x=\sin\theta \Rightarrow dx=\cos\theta\ d\theta$

$$=\int \sin^2\theta\ \sqrt{1-\ \sin^2\theta}\ \ \cos\theta\ \ d\theta=\int \sin^2\theta\ \ \ \cos^2\theta\ \ d\theta$$

$$=\frac{1}{4}\int 4\ \sin^2\theta\ \ \ \cos^2\theta\ \ d\theta=\frac{1}{4}\int(2\sin\theta\cos\theta)^2\ \ d\theta=\frac{1}{4}\int \sin^2 2\theta\ \ d\theta$$

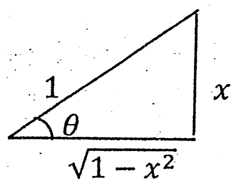
$$=\frac{1}{4}\int\frac{1-\cos 4\theta}{2}\ d\theta=\frac{1}{8}\left[\theta-\frac{1}{4}\sin 4\theta\right]+c=\frac{1}{8}\left[\theta-\frac{1}{4}(2\sin 2\theta\cos 2\theta)\right]+c$$

$$=\frac{1}{8}\left[\theta-\frac{1}{4}\left(4\sin\theta\cos\theta\left(1-2\sin^2\theta\right)\right)\right]+c$$

$\because x=\sin\theta \Rightarrow \theta=\sin^{-1}x$

$$=\frac{1}{8}[\theta-\sin\theta\cos\theta\,(1-2\sin^2\theta)]+c$$

$\cos\theta=\sqrt{1-x^2}$



$$=\frac{1}{8}\left[\sin^{-1}x-x\sqrt{1-x^2}(1-2x^2)\right]+c$$

5) $\int\frac{\sqrt{1-x^2}}{x}\ \ dx$ $\qquad$ *let* $x=\sin\theta \Rightarrow dx=\cos\theta\ d\theta$

$$=\int\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cos\theta\, d\theta=\int\frac{\cos\theta}{\sin\theta}\ \cos\theta\, d\theta=\int\frac{\cos^2\theta}{\sin\theta}\ d\theta$$

$$=\int\frac{1-\sin^2\theta}{\sin\theta}\ d\theta=\int(\csc\theta-\sin\theta)d\theta=\ln|\csc\theta-\cot\theta|+\cos\theta+c$$

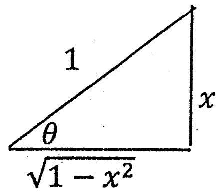
$$=\ln\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right|+\sqrt{1-x^2}+c$$

$\because x=\sin\theta$

$\cot\theta=\frac{\sqrt{1-x^2}}{x}$

$\csc\theta=\frac{1}{x}$

$$=\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|+\sqrt{1-x^2}+c$$



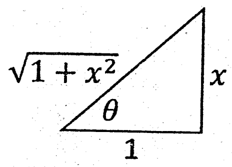
6) $\int \frac{\sqrt{1+x^2}}{x}\, dx$ $\qquad$ *let* $x=\tan\theta \Rightarrow dx=\sec^2\theta\ d\theta$

$$=\int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\sec^2\theta\ d\theta=\int\frac{\sec\theta}{\tan\theta}\,(1+\tan^2\theta)\, d\theta$$

$$=\int\left(\frac{\sec\theta}{\tan\theta}+\sec\theta\ \tan\theta\right)d\theta=\int(\csc\theta-\sec\theta\ \tan\theta)d\theta$$

$$=\ln|\csc\theta-\cot\theta|+\sec\theta+c$$

$\because x=\tan\theta$



$$=\ln\left|\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right|+\sqrt{1+x^2}+c$$

$$=\ln\left|\frac{\sqrt{1+x^2}-1}{x}\right|+\sqrt{1+x^2}+c$$

7) $\int\frac{x^2}{\sqrt{4-x^2}}\, dx$ $\qquad$ *let* $x=2\sin\theta \Rightarrow dx=2\cos\theta\ d\theta$

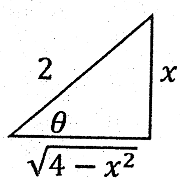
$$=\int\frac{4\sin^2\theta\ 2\cos\theta\ d\theta}{\sqrt{4-4\sin^2\theta}}=\int\frac{4\sin^2\theta\ 2\cos\theta\ d\theta}{2\cos\theta}=\int 4\sin^2\theta\ d\theta$$

$$=\int 4\times\frac{1}{2}(1-\cos 2\theta)\ d\theta=2\int(1-\cos 2\theta)\ d\theta$$

$$=2\left(\theta-\frac{1}{2}\sin 2\theta\right)+c\ =2\left(\theta-\frac{1}{2}\times 2\ \sin\theta\ \cos\theta\right)+c$$

$$=2\,\theta-2\ \sin\theta\ \cos\theta+c$$

$\because x=2\sin\theta\Rightarrow\sin\theta=\frac{x}{2}$



$$=2\sin^{-1}\frac{x}{2}+2\times\frac{x}{2}\frac{\sqrt{4-x^2}}{2}+c$$

$$=2\sin^{-1}\frac{x}{2}+\frac{1}{2}x\sqrt{4-x^2}+c$$

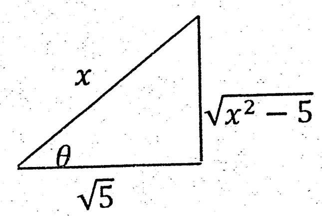
8) $\int \frac{\sqrt{x^2-5}}{x}\ dx$ $\qquad$ $let\ \ x=\sqrt{5}\sec\theta \Longrightarrow dx=\sqrt{5}\sec\theta\tan\theta\ \ d\theta$

$$=\int \frac{\sqrt{5\sec^2\theta-5}}{\sqrt{5}\sec\theta}\ \sqrt{5}\sec\theta\tan\theta\ \ d\theta=\sqrt{5}\int \tan^2\theta\quad d\theta$$

$$=\sqrt{5}\int(\sec^2\theta-1)\quad d\theta=\sqrt{5}\,(\tan\theta-\ \theta)+c$$

$\because x=\sqrt{5}\sec\theta \Longrightarrow \sec\theta=\frac{x}{\sqrt{5}}$

$$=\sqrt{5}\left(\frac{\sqrt{x^2-5}}{\sqrt{5}}-\ \sec^{-1}\frac{x}{\sqrt{5}}\right)+c$$



$$=\sqrt{x^2-5}-\ \sqrt{5}\sec^{-1}\frac{x}{\sqrt{5}}+c$$

9) $\int \frac{\sqrt{9+x^2}}{x^4}\ dx$ $\qquad$ $let\ \ x=3\tan\theta \Longrightarrow dx=3\sec^2\theta\, d\theta$

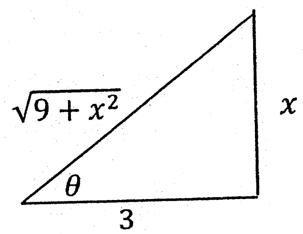
$$=\int \frac{\sqrt{9+9\tan^2\theta}}{81\tan^4\theta}\ 3\sec^2\theta\, d\theta=\int \frac{3\sec\theta}{81\tan^4\theta}\ 3\sec^2\theta\, d\theta=\frac{1}{9}\int \frac{\sec^3\theta}{\tan^4\theta}\ d\theta$$

$$=\frac{1}{9}\int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^4\theta}{\cos^4\theta}}\ d\theta=\frac{1}{9}\int \frac{\cos\theta}{\sin^4\theta}\ d\theta=\frac{1}{9}\int(\sin\theta)^{-4}\cos\theta\ \ d\theta$$

$$=-\frac{1}{27}\,\frac{1}{\sin^3\theta}+c=-\frac{1}{27}\csc^3\theta+c$$

$\because x=3\tan\theta \Longrightarrow \tan\theta=\frac{x}{3}$

$$=-\frac{1}{27}\left(\frac{\sqrt{9+x^2}}{x}\right)^3+c$$



10) $\int \frac{dx}{(1+2x^2)^{5/2}}$

let $\sqrt{2}\,x = \tan\theta \Longrightarrow \sqrt{2}\,dx = \sec^2\theta\, d\theta$

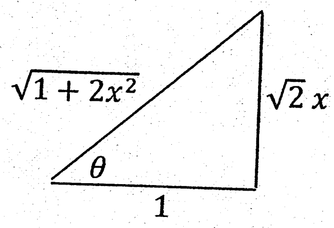
$$= \frac{1}{\sqrt{2}}\int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{5/2}} = \frac{1}{\sqrt{2}}\int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{5/2}} = \frac{1}{\sqrt{2}}\int \frac{\sec^2\theta\, d\theta}{\sec^5\theta}$$

$$= \frac{1}{\sqrt{2}}\int \cos^3\theta\ d\theta = \frac{1}{\sqrt{2}}\int \cos^2\theta\cos\theta\ d\theta$$

$$= \frac{1}{\sqrt{2}}\int (1-\sin^2\theta)\cos\theta\ d\theta = \frac{1}{\sqrt{2}}\int (\cos\theta - \sin^2\theta\ \cos\theta)\ d\theta$$

$$= \frac{1}{\sqrt{2}}\left(\sin\theta + \frac{\sin^3\theta}{3}\right) + c$$

$\because \sqrt{2}\,x = \tan\theta$



$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}\,x}{\sqrt{1+2x^2}} + \frac{1}{3}\left(\frac{\sqrt{2}\,x}{\sqrt{1+2x^2}}\right)^3\right) + c$$

11) $\int x\sqrt{1-x^4}\ dx$

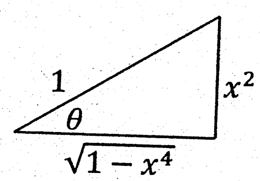
let $x^2 = \sin\theta \Longrightarrow 2x\,dx = \cos\theta\ d\theta$

$$= \int \sqrt{1-(x^2)^2}\ \ x\,dx = \int \sqrt{1-(\sin\theta)^2}\ \ \frac{\cos\theta}{2}d\theta = \frac{1}{2}\int \cos^2\theta\ d\theta$$

$$= \frac{1}{2}\int \frac{1+\cos 2\theta}{2}\ d\theta = \frac{1}{4}\int 1+\cos 2\theta\ d\theta = \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right] + c$$

$$= \frac{1}{4}[\theta + \sin\theta\cos\theta\,] + c$$

$\because\ x^2 = \sin\theta$



$$= \frac{1}{4}[\sin^{-1}x^2 + \sin\theta\cos\theta\,] + c$$

12) $\displaystyle\int \frac{dx}{(4x-x^2)^{3/2}}$

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4 - 4)$$

$$= -[(x-2)^2 - 4] = [4 - (x-2)^2]$$

$$let\ \ (x-2) = 2\sin\theta \ ,\ \ dx = 2\cos\theta\, d\theta$$

$$\therefore \int \frac{dx}{(4x-x^2)^{3/2}} = \int \frac{dx}{[4-(x-2)^2]^{3/2}} = \int \frac{2\cos\theta \quad d\theta}{[4-4\sin^2\theta]^{3/2}}$$

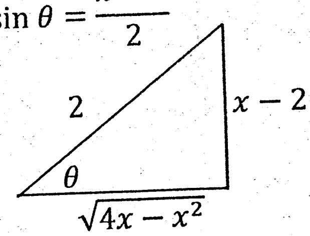
$$= 2\int \frac{\cos\theta \quad d\theta}{[4\cos^2\theta]^{3/2}} = \frac{2}{8}\int \frac{\cos\theta \quad d\theta}{\cos^3\theta}$$

$$= \frac{1}{4}\int \sec^2\theta \quad d\theta = \frac{1}{4}\tan\theta + c$$

$$= \frac{1}{4}\frac{x-2}{\sqrt{4x-x^2}} + c$$

$$\because (x-2) = 2\sin\theta$$

$$\Longrightarrow \sin\theta = \frac{x-2}{2}$$



13) $\displaystyle\int \frac{x^2\, dx}{\sqrt{4x-x^2}}$

من المثال السابق : $x - 2 = 2\sin\theta$

$$\therefore x = 2\sin\theta + 2$$

$$= \int \frac{x^2\, dx}{\sqrt{4-(x-2)^2}} = \int \frac{(2\sin\theta+2)^2\, 2\cos\theta \quad d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$= \int (2\sin\theta + 2)^2 \quad d\theta = \int (4\sin^2\theta + 8\sin\theta + 4)\, d\theta$$

$$= \int \left(4\cdot\frac{1}{2}(1-\cos 2\theta) + 8\sin\theta + 4\right) d\theta = \int (-2\cos 2\theta + 8\sin\theta + 6)\, d\theta$$

$$= 6\theta - \sin 2\theta - 8\cos\theta + c = 6\theta - 2\sin\theta\ \cos\theta - 8\cos\theta + c$$

$$= 6\ \sin^{-1}\left(\frac{x-2}{2}\right) - 2\cdot\frac{x-2}{2}\,\frac{\sqrt{4x-x^2}}{2} - 8\frac{\sqrt{4x-x^2}}{2} + c$$

$$= 6\ \sin^{-1}\left(\frac{x-2}{2}\right) - \frac{(x-2)\sqrt{4x-x^2}}{2} - 4\sqrt{4x-x^2} + c$$

14) $\int \frac{dx}{(x^2+9)^2}$ let $x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta \; d\theta$

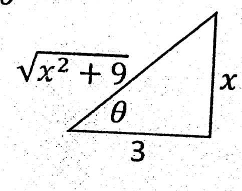
$$= \int \frac{3\sec^2\theta \; d\theta}{(9\tan^2\theta+9)^2} = \int \frac{3\sec^2\theta \; d\theta}{81(\tan^2\theta+1)^2} = \frac{1}{27}\int \frac{\sec^2\theta \; d\theta}{(\sec^2\theta)^2}$$

$$= \frac{1}{27}\int \cos^2\theta \; d\theta = \frac{1}{27}\int \frac{1}{2}(1+\cos 2\theta)d\theta = \frac{1}{54}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{54}(\theta + \sin\theta \; \cos\theta) + C$$

$\because x = 3\tan\theta$

$$= \frac{1}{54}\left(\tan^{-1}\frac{x}{3} + \frac{x}{\sqrt{x^2+9}}\frac{3}{\sqrt{x^2+9}}\right) + c$$



$$= \frac{1}{54}\left(\tan^{-1}\frac{x}{3} + \frac{3x}{x^2+9}\right) + c$$

15) $\int \frac{x^2}{(4-x^2)^{\frac{5}{2}}} \; dx$ let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta \; d\theta$

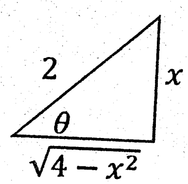
$$= \int \frac{4\sin^2\theta \; 2\cos\theta \; d\theta}{(4-4\sin^2\theta)^{\frac{5}{2}}} = \int \frac{8\sin^2\theta \; \cos\theta \; d\theta}{4^{\frac{5}{2}}(1-\sin^2\theta)^{\frac{5}{2}}} = \int \frac{8\sin^2\theta \; \cos\theta \; d\theta}{32\cos^5\theta}$$

$$= \frac{1}{4}\int \frac{\sin^2\theta \; d\theta}{\cos^4\theta} \; d\theta = \frac{1}{4}\int \frac{\sin^2\theta \; d\theta}{\cos^2\theta \; \cos^2\theta} = \frac{1}{4}\int \tan^2\theta \sec^2\theta \; d\theta$$

$$= \frac{1}{4}\frac{\tan^3\theta}{3} + c = \frac{1}{12}\tan^3\theta + C$$

$\because x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$

$$= \frac{1}{12}\left(\frac{x}{\sqrt{4-x^2}}\right)^3 + C$$



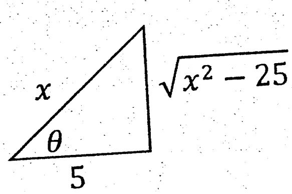
16) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-25}}$ $\qquad$ $let\ \ x = 5\sec\theta \Rightarrow dx = 5\sec\theta\tan\theta\ d\theta$

$$= \int \frac{5\sec\theta\ \tan\theta\ d\theta}{25\sec^2\theta\ \sqrt{25\sec^2\theta - 25}} = \int \frac{5\sec\theta\ \tan\theta\ d\theta}{25\sec^2\theta\ 5\tan\theta}$$

$$= \frac{1}{25}\int \frac{1}{\sec\theta}\ d\theta = \frac{1}{25}\int \cos\theta\ d\theta = \frac{1}{25}\ \sin\theta + C$$

$\because x = 5\sec\theta$

$$= \frac{1}{25} \times \frac{\sqrt{x^2-25}}{x} + c = \frac{\sqrt{x^2-25}}{25x} + C \qquad \sec\theta = \frac{x}{5}$$



17) $\displaystyle\int \frac{(16-9x^2)^{\frac{3}{2}}}{x^6}\ dx = \int \frac{((4)^2-(3x)^2)^{\frac{3}{2}}}{x^6}\ dx$ $\qquad$ $let\ \ 3x = 4\sin\theta \Rightarrow 3dx = 4\cos\theta\ d\theta$

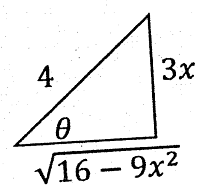
$$\Rightarrow dx = \frac{4}{3}\cos\theta\ d\theta$$

$$= \int \frac{(16-16\sin^2\theta)^{\frac{3}{2}}\ \frac{4}{3}\cos\theta\ d\theta}{\left(\frac{4}{3}\right)^6\ \sin^6\theta}$$

$$= \left(\frac{3}{4}\right)^6\left(\frac{4}{3}\right)(16)^{\frac{3}{2}}\int \frac{(1-\ \sin^2\theta)^{\frac{3}{2}}\ \cos\theta\ d\theta}{\sin^6\theta} = \frac{243}{16}\int \frac{\cos^4\theta\ \ d\theta}{\sin^6\theta}$$

$$= \frac{243}{16}\int \cot^4\theta\csc^2\theta\ \ d\theta = -\frac{243}{16} \times \frac{\cot^5\theta}{5} + C$$

$$= -\frac{243}{80}\cot^5\theta + C\ = -\frac{243}{80}\left(\frac{\sqrt{16-9x^2}}{3x}\right)^5 + C \qquad \because 3x = 4\sin\theta \Rightarrow \sin\theta = \frac{3x}{4}$$



$$= -\frac{243}{80} \times \frac{(16-9x^2)^{\frac{5}{2}}}{243\ x^5} + C$$

$$= -\frac{1}{80}(16-9x^2)^{\frac{5}{2}} + C$$

$$18)\int \frac{dy}{y(2\,y^3+1)^2}$$

$$let\ \ 2y^3=\tan^2\theta \implies 3y^2\,dy=\tan\theta\sec^2\theta\ d\theta$$

$$\implies 3y^2\,dy=\tan\theta\sec^2\theta\ d\theta$$

$$\because y^3=\frac{1}{2}\tan^2\theta \Rightarrow y=\frac{1}{\sqrt[3]{2}}(\tan\theta)^{\frac{2}{3}}$$

$$\implies 3\left(\frac{1}{\sqrt[3]{2}}(\tan\theta)^{\frac{2}{3}}\right)^2\ dy=\tan\theta\sec^2\theta\ d\theta$$

$$\implies dy=\frac{\sqrt[3]{4}\ \tan\theta\sec^2\theta}{3(\tan\theta)^{\frac{4}{3}}}\ d\theta=\frac{\sqrt[3]{4}\ \ \sec^2\theta}{3(\tan\theta)^{\frac{1}{3}}}\ d\theta$$

$$=\int\frac{\dfrac{\sqrt[3]{4}\ \ \sec^2\theta}{3(\tan\theta)^{\frac{1}{3}}}\ d\theta}{\dfrac{1}{\sqrt[3]{2}}(\tan\theta)^{\frac{2}{3}}(\tan^2\theta+1)^2}$$

$$=\frac{\sqrt[3]{4}}{3}\times\sqrt[3]{2}\int\frac{\sec^2\theta\ \ d\theta}{(\tan\theta)^{\frac{1}{3}}\ (\tan\theta)^{\frac{2}{3}}\ \sec^4\theta}$$

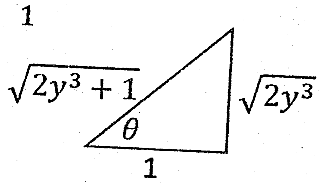
$$=\frac{\sqrt[3]{8}}{3}\int\frac{d\theta}{\tan\theta\ \ \sec^2\theta}=\frac{2}{3}\int\frac{d\theta}{\tan\theta\ \ \sec^2\theta}$$

$$\frac{\cos}{\sin}$$

$$=\frac{2}{3}\int\frac{\cos\theta\ \ \cos^2\theta\ \ d\theta}{\sin\theta}=\frac{2}{3}\int\frac{\cos\theta\ \ (1-\sin^2\theta)\ d\theta}{\sin\theta}$$

$$=\frac{2}{3}\int(\cot\theta-\sin\theta\cos\theta)\ d\theta$$

$$\because \tan\theta=\frac{\sqrt{2y^3}}{1}$$



$$=\frac{2}{3}\left[\ln|\sin\theta|-\frac{1}{2}\sin^2\theta\right]+C$$

$$=\frac{2}{3}\left[\ln\left|\frac{\sqrt{2y^3}}{\sqrt{2y^3+1}}\right|-\frac{1}{2}\left(\frac{\sqrt{2y^3}}{\sqrt{2y^3+1}}\right)^2\right]+C$$

$$=\frac{2}{3}\left[\ln\left|\sqrt{\frac{2y^3}{2y^3+1}}\right|-\frac{1}{2}\frac{2y^3}{2y^3+1}\right]+C$$

$$=\frac{2}{3}\left[\frac{1}{2}\ln\left|\frac{2y^3}{2y^3+1}\right|-\frac{1}{2}\frac{2y^3}{2y^3+1}\right]+C$$

$$=\frac{1}{3}\left[\ln\left|\frac{2y^3}{2y^3+1}\right|-\frac{2y^3}{2y^3+1}\right]+C$$

$-\frac{\sin}{\cos}$ $-\ln\cos$ $\ln\sec$

19) $\int_0^1 \frac{dx}{x+\sqrt{x}+1}$

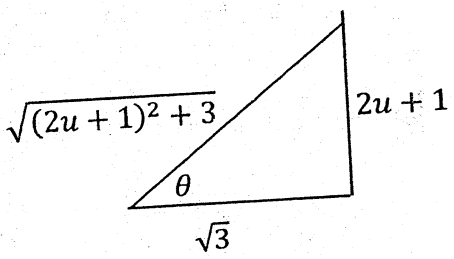
let $x = u^2 \Rightarrow dx = 2u\,du$

$$u^2+u+1$$
$$= u^2+u+\frac{1}{4}-\frac{1}{4}+1$$
$$= \left(u+\frac{1}{2}\right)^2+\frac{3}{4}$$

let $u+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$

$$\Rightarrow u = \frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}$$
$$\Rightarrow du = \frac{\sqrt{3}}{2}\sec^2\theta\ d\theta$$
$$x=0 \Rightarrow u=0 \Rightarrow \frac{\sqrt{3}}{2}\tan\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$
$$x=1 \Rightarrow u=1 \Rightarrow \frac{\sqrt{3}}{2}\tan\theta = \frac{3}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\because\ \tan\theta = \frac{2u+1}{\sqrt{3}}$$



$$I = \int \frac{2u\ du}{u^2+u+1} = \int \frac{2u\ du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\left(\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}\right)\ \frac{\sqrt{3}}{2}\sec^2\theta\ d\theta}{\left(\frac{\sqrt{3}}{2}\tan\theta\right)^2+\frac{3}{4}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\left(\sqrt{3}\tan\theta-1\right)\ \frac{\sqrt{3}}{2}\sec^2\theta\ d\theta}{\frac{3}{4}\tan^2\theta+\frac{3}{4}}$$

$$= \frac{\sqrt{3}}{2}\times\frac{4}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\left(\sqrt{3}\tan\theta-1\right)\ \sec^2\theta\ d\theta}{\tan^2\theta+1}$$

$$= \frac{2\sqrt{3}}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\left(\sqrt{3}\tan\theta-1\right)\ \sec^2\theta\ d\theta}{\sec^2\theta}$$

$$= \frac{2\sqrt{3}}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sqrt{3}\tan\theta-1\right)\ d\theta$$

$$= \frac{2\sqrt{3}}{3}\left[\sqrt{3}\ \ln|\sin\theta|-\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{2\sqrt{3}}{3}\left[\left(\sqrt{3}\ \ln\left|\sin\frac{\pi}{3}\right|-\frac{\pi}{3}\right)-\left[\sqrt{3}\ \ln\left|\sin\frac{\pi}{6}\right|-\frac{\pi}{6}\right]\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{2\sqrt{3}}{3}\left[\left(\sqrt{3}\ \ln\frac{\sqrt{3}}{2}-\frac{\pi}{3}\right)-\left[\sqrt{3}\ \ln\frac{1}{2}-\frac{\pi}{6}\right]\right] = \frac{2\sqrt{3}}{3}\left[\sqrt{3}\ \ln\frac{\sqrt{3}}{2}-\frac{\pi}{3}-\sqrt{3}\ \ln\frac{1}{2}+\frac{\pi}{6}\right]$$

$$= \frac{2\sqrt{3}}{3}\left[\sqrt{3}\ \left(\ln\frac{\sqrt{3}}{2}-\ \ln\frac{1}{2}\right)-\frac{\pi}{6}\right] = \frac{2\sqrt{3}}{3}\left[\sqrt{3}\ \ln\sqrt{3}-\frac{\pi}{6}\right]$$

$$20)\ \int \frac{(2e^{2x}-e^{x})\,dx}{\sqrt{3e^{2x}-6e^{x}-1}}$$

$$let\ \ u = e^{x} \Rightarrow du = e^{x}\,dx$$

$$= \int \frac{(2e^{x}-1)\,e^{x}\,dx}{\sqrt{3e^{2x}-6e^{x}-1}} = \int \frac{(2u-1)\;du}{\sqrt{3u^{2}-6u-1}}$$

$$3u^{2}-6u-1$$

$$= 3\left[u^{2}-2u-\frac{1}{3}\right]$$

$$= \ 3\left[u^{2}-2u+1-1-\frac{1}{3}\right]$$

$$= \ 3\left[(u-1)^{2}-\frac{4}{3}\right]$$

$$= \int \frac{(2u-1)\,du}{\sqrt{3\left[(u-1)^{2}-\frac{4}{3}\right]}}$$

$$let\ \ u-1 = \frac{2}{\sqrt{3}}\sec\theta$$

$$\Rightarrow u = \frac{2}{\sqrt{3}}\sec\theta + 1$$

$$\Rightarrow du = \frac{2}{\sqrt{3}}\sec\theta\tan\theta\ d\theta$$

$$= \frac{1}{\sqrt{3}}\int \frac{\left[2\left(\frac{2}{\sqrt{3}}\sec\theta+1\right)-1\right]\ \frac{2}{\sqrt{3}}\sec\theta\tan\theta\ d\theta}{\sqrt{\frac{4}{3}\sec^{2}\theta-\frac{4}{3}}}$$

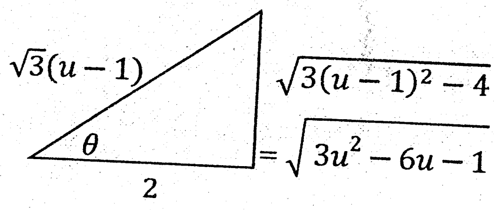
$$= \frac{1}{\sqrt{3}}\int \frac{\left[\frac{4}{\sqrt{3}}\sec\theta+2-1\right]\frac{2}{\sqrt{3}}\ \sec\theta\tan\theta\ d\theta}{\sqrt{\frac{4}{3}(\sec^{2}\theta-1)}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{\left[\frac{4}{\sqrt{3}}\sec\theta+1\right]\frac{2}{\sqrt{3}}\ \sec\theta\tan\theta\ d\theta}{\frac{2}{\sqrt{3}}\tan\theta}$$

$$= \frac{1}{\sqrt{3}}\int \left[\frac{4}{\sqrt{3}}\sec\theta+1\right]\sec\theta\ d\theta$$

$$\because\ \ \sec\theta = \frac{\sqrt{3}(u-1)}{2}$$

$$= \frac{1}{\sqrt{3}}\int \left[\frac{4}{\sqrt{3}}\sec^{2}\theta+\sec\theta\right]\ d\theta$$



$$= \frac{1}{\sqrt{3}}\left[\frac{4}{\sqrt{3}}\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right]+C$$

$$= \frac{1}{\sqrt{3}}\left[\frac{4}{\sqrt{3}}\frac{\sqrt{3u^{2}-6u-1}}{2}+\ln\left|\frac{\sqrt{3}(u-1)}{2}+\frac{\sqrt{3u^{2}-6u-1}}{2}\right|\ \right]+C$$

$$= \frac{1}{\sqrt{3}}\left[\frac{2}{\sqrt{3}}\sqrt{3\,e^{2x}-6e^{x}-1}+\ln\left|\frac{\sqrt{3}}{2}(e^{x}-1)+\frac{1}{2}\sqrt{3\,e^{2x}-6e^{x}-1}\right|\ \right]+C$$

$$21)\int \frac{\sec^2 x\ dx}{\tan x\sqrt{8\tan x-\tan^2 x}}$$

$$y=\tan x \ \Rightarrow dy=\sec^2 x\ dx$$

$$8y-y^2=-[y^2-8y]$$

$$=-[y^2-8y+16-16]$$

$$=-[(y-4)^2-16]$$

$$=[16-(y-4)^2]$$

$$y-4=4\sin\theta \Longrightarrow y=4+4\sin\theta$$

$$\Rightarrow dy=4\cos\theta\ d\theta$$

$$I\ =\int\frac{dy}{y\sqrt{8y-y^2}}$$

$$=\int\frac{dy}{y\sqrt{[16-(y-4)^2]}}$$

$$=\int\frac{4\cos\theta\ d\theta}{(4+4\sin\theta)\sqrt{[16-16\ \sin^2 x]}}$$

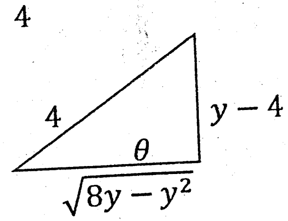
$$=\int\frac{4\cos\theta\ d\theta}{(4+4\sin\theta)4\cos\theta}$$

$$=\frac{1}{4}\int\frac{d\theta}{(1+\sin\theta)}$$

$$=\frac{1}{4}\int\frac{(1-\sin\theta)\ d\theta}{(1-\sin^2\theta)}=\frac{1}{4}\int\frac{(1-\sin\theta)\ d\theta}{\cos^2\theta}=\frac{1}{4}\int\left(\frac{1}{\cos^2\theta}-\frac{\sin\theta}{\cos^2\theta}\right)d\theta$$

$$=\frac{1}{4}\int(\sec^2\theta-\sec\theta\tan\theta)\ d\theta=\frac{1}{4}(\tan\theta-\sec\theta)+c$$

$$\because \sin\theta=\frac{y-4}{4}$$



$$=\frac{1}{4}\left(\frac{y-4}{\sqrt{8y-y^2}}-\frac{4}{\sqrt{8y-y^2}}\right)+c$$

$$=\frac{1}{4}\left(\frac{y-8}{\sqrt{8y-y^2}}\right)+c$$

$$\therefore\int\frac{\sec^2 x\ dx}{\tan x\sqrt{8\tan x-\tan^2 x}}=\frac{1}{4}\left(\frac{\tan x-8}{\sqrt{8\tan x-\tan^2 x}}\right)+c$$

$$22)\ \int \frac{\sin^2 x\ \cos x\ \ dx}{\sqrt{\sin^2 x + 4}}$$

$$I = \int \frac{y^2\, dy}{\sqrt{y^2 + 4}}$$

$$y = \sin x \ \Rightarrow dy = \cos x\ dx$$

$$= \int \frac{4\tan^2\theta \quad 2\sec^2\theta\, d\theta}{\sqrt{4\tan^2\theta + 4}}$$

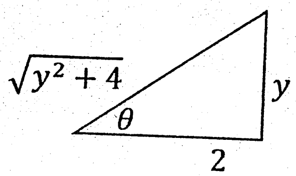
$$y = 2\tan\theta \Rightarrow dy = 2\sec^2\theta\ d\theta$$

$$= \int \frac{4\tan^2\theta \quad 2\sec^2\theta\, d\theta}{2\sec\theta} = 4\int \tan^2\theta\ \sec\theta\, d\theta$$

$$= 4\left(\frac{1}{2}\tan\theta\sec\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\ \right) + c$$

$$\because \tan\theta = \frac{y}{2}$$

$$= 4\left[\frac{1}{2}\times\frac{y}{2}\times\frac{\sqrt{y^2+4}}{2} - \frac{1}{2}\ln\left|\frac{\sqrt{y^2+4}}{2} + \frac{y}{2}\right|\ \right] + c$$



$$= \frac{1}{2}y\sqrt{y^2+4} - 2\ln\left|\frac{\sqrt{y^2+4} + y}{2}\right| + c$$

$$\therefore \int \frac{\sin^2 x\ \cos x\ \ dx}{\sqrt{\sin^2 x + 4}} = \frac{1}{2}\sin x\ \sqrt{\sin^2 x + 4} - 2\ln\left|\frac{\sqrt{\sin^2 x + 4} + \sin x}{2}\right| + c$$

$$23)\int \frac{\cos x \ \ dx}{\sin x\sqrt{4+\sin x}} \qquad\qquad y=\sin x \ \Rightarrow dy=\cos x \ \ dx$$

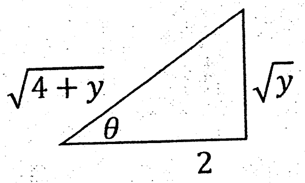
$$I=\int \frac{dy}{y\sqrt{4+y}} \qquad\qquad \sqrt{y}=2\tan\theta \Longrightarrow \frac{1}{2\sqrt{y}}dy=2\sec^2\theta \ \ d\theta$$

$$=\int \frac{8\ \tan\theta \ \sec^2\theta\ d\theta}{4\tan^2\theta\ \sqrt{4+4\tan^2\theta}} \qquad\qquad \Rightarrow dy=4\sqrt{y}\ \sec^2\theta \ \ d\theta$$

$$\Rightarrow dy=8\ \tan\theta\ \sec^2\theta\ \ d\theta$$

$$=\int \frac{8\ \tan\theta\ \sec^2\theta\, d\theta}{4\tan^2\theta\ \ 2\sec\theta}=\int \frac{\sec\theta\ \ d\theta}{\tan\theta}=\int \frac{1\ d\theta}{\sin\theta}$$

$$=\int \csc\theta\ \ d\theta \ =\ln|\csc\theta-\cot\theta|+c \qquad\qquad \because \tan\theta=\frac{y}{2}$$



$$=\ln\left|\frac{\sqrt{4+y}}{\sqrt{y}}-\frac{2}{\sqrt{y}}\right|+c=\ln\left|\frac{\sqrt{4+y}-2}{\sqrt{y}}\right|+c$$

$$\therefore \int \frac{\cos x \ \ dx}{\sin x\sqrt{4+\sin x}} \ =\ln\left|\frac{\sqrt{4+\sin x}-2}{\sqrt{\sin x}}\right|+c$$

$$24)\int \frac{\sin 2x \ \ dx}{\sqrt{1+\cos x}}=\int \frac{2\ \sin x\cos x \ \ dx}{\sqrt{1+\cos x}} \qquad\qquad y=\cos x \ \Rightarrow dy=-\sin x \ \ dx$$

$$=\int \frac{-2y\ \ dy}{\sqrt{1+y}}=-2\int y\,(1+y)^{-\frac{1}{2}}dy \qquad\qquad u=1+y\Rightarrow y=u-1$$

$$\Rightarrow dy=du$$

$$=-2\int (u-1)\,u^{-\frac{1}{2}}du=-2\int \left(u^{\frac{1}{2}}-u^{-\frac{1}{2}}\right)\ du$$

$$=-2\left(\frac{2\,u^{\frac{3}{2}}}{3}-2\,u^{\frac{1}{2}}\right)+c=-4\,u^{\frac{1}{2}}\left(\frac{u}{3}-1\right)=4\sqrt{u}\left(1-\frac{u}{3}\right)=\frac{4}{3}\sqrt{u}(\,3-u)$$

$$u=1+y\Rightarrow \int \frac{-2y\ \ dy}{\sqrt{1+y}}=\frac{4}{3}\sqrt{1+y}(\,3-(1+y))=\frac{4}{3}(2-y)\sqrt{1+y}$$

$$y=\cos x \ \ \Rightarrow \int \frac{\sin 2x \ \ dx}{\sqrt{1+\cos x}}=\frac{4}{3}(2-\cos x)\sqrt{1+\cos x}$$

التعويض $\left(z = \tan\frac{x}{2}\right)$:

$$\cos x = \frac{1-z^2}{1+z^2} \quad , \sin x = \frac{2z}{1+z^2} \quad , \tan x = \frac{2z}{1-z^2} \quad , dz = \frac{2dz}{1+z^2}$$

الإثبات :

$$1) \quad \cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{\sec^2\frac{x}{2}} - 1$$

$$= \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+z^2} - 1 = \frac{2-1-z^2}{1+z^2} = \frac{1-z^2}{1+z^2}$$

$$2) \quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\cdot\frac{1}{\sec^2\frac{x}{2}}$$

$$= \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2z}{1+z^2}$$

$$3) \quad z = \tan\frac{x}{2} \Rightarrow \frac{x}{2} = \tan^{-1} z \Rightarrow x = 2\tan^{-1} z \Rightarrow dx = \frac{2\,dz}{1+z^2}$$

أمثلة /

$$1) \int \frac{1}{1+\cos x}dx = \int \frac{1}{1+\frac{1-z^2}{1+z^2}}\,\frac{2\,dz}{1+z^2} = \int \frac{1+z^2}{2}\cdot\frac{2\,dz}{1+z^2} = \int dz = z + c$$

$$= \tan\left(\frac{x}{2}\right) + c$$

$$2)\int \frac{1}{2+\sin x}dx = \int \frac{1}{2+\frac{2z}{1+z^2}}\ \frac{2\,dz}{1+z^2} = \int \frac{1+z^2}{2z^2+2z+2}\ \frac{2\,dz}{1+z^2}$$

$$= \int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+1/2)^2+3/4} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{z+1/2}{\sqrt{3}/4}\right)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan(x/2)+1}{\sqrt{3}}\right)+c$$

$$3)\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}$$

$$\because z=\tan\frac{\theta}{2}\ ,\theta=\frac{\pi}{2} \Rightarrow z=\tan\frac{\pi}{4}=1$$

$$,\theta=0 \Rightarrow z=\tan 0=0$$

$$= \int_0^1 \frac{\frac{2dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} = \int_0^1 \frac{2dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2dz}{z^2+3}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}+c = \frac{\pi}{3\sqrt{3}}+c$$

$$4)\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta}$$

$$\because z=\tan\frac{\theta}{2}\ ,\theta=\frac{2\pi}{3} \Rightarrow z=\tan\frac{2\pi}{3}=\sqrt{3}$$

$$,\theta=\frac{\pi}{2} \Rightarrow z=\tan\frac{\pi}{4}=1$$

$$= \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{\cos\theta\,d\theta}{\sin\theta(\cos\theta+1)} = \int_{\pi/2}^{2\pi/3} \frac{\cot\theta\,d\theta}{(\cos\theta+1)} = \int_1^{\sqrt{3}} \frac{\frac{1-z^2}{2z}\ \frac{2dz}{1+z^2}}{\frac{1-z^2}{1+z^2}+1}$$

$$= \int_1^{\sqrt{3}} \frac{1-z^2}{2z}dz = \frac{1}{2}\int_1^{\sqrt{3}}\left(\frac{1}{z}-z\right)dz = \frac{1}{2}\left[\ln z-\frac{z^2}{2}\right]_1^{\sqrt{3}} = \frac{1}{2}\left[\left(\ln\sqrt{3}-\frac{3}{2}\right)-\left(\ln 1-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\left(\ln\sqrt{3}-\frac{3}{2}\right)-\left(0-\frac{1}{2}\right)\right] = \frac{1}{2}(\ln\sqrt{3}-1)$$

5) $\int \frac{dx}{3+5\sin x+3\cos x}$

$$= \int \frac{\frac{2dz}{1+z^2}}{3+5\cdot\frac{2z}{1+z^2}+3\cdot\frac{1-z^2}{1+z^2}} = \int \frac{2dz}{3+3z^2+10z+3-3z^2}$$

$$= \int \frac{2\,dz}{6+10z} = \int \frac{dz}{3+5\,z} = \frac{1}{5}\ln|3+5\,z|+c$$

$$\therefore \int \frac{dx}{3+5\sin x+3\cos x} = \frac{1}{5}\ln\left|3+5\tan\frac{x}{2}\right|+c$$

6) $\int \frac{dx}{5+\sin x+3\cos x}$

$$= \int \frac{\frac{2dz}{1+z^2}}{5+\frac{2z}{1+z^2}+3\cdot\frac{1-z^2}{1+z^2}} = \int \frac{2dz}{5+5z^2+2z+3-3z^2}$$

$$= \int \frac{2\,dz}{2z^2+2z+8} = \int \frac{dz}{z^2+z+4} = \int \frac{dz}{z^2+z+\frac{1}{4}-\frac{1}{4}+4}$$

$$= \int \frac{dz}{z^2+z+\frac{1}{4}+\frac{15}{4}} = \int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\frac{15}{4}} = \frac{2}{\sqrt{15}}\tan^{-1}\frac{2\left(z+\frac{1}{2}\right)}{\sqrt{15}}+c$$

$$= \frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{2z+1}{\sqrt{15}}\right)+c$$

$$\therefore \int \frac{dx}{5+\sin x+3\cos x} = \frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{2\left(\tan\frac{x}{2}\right)+1}{\sqrt{15}}\right)+c$$

التكامل المعتل

لكى نستطيع إيجاد $\int_a^b f(x)\,dx$ لابد الفترة من a إلى b مغلقة ومحدودة ولكن الكثير من الدوال تواجهنا والفترة المراد التكامل عليها إما أن تكون فترة غير مغلقة مثل $[a ، \infty)$ ، $(-\infty، b]$ ، $(-\infty، \infty)$ ويسمى التكامل في هذه الحالة بالتكامل اللانهائي ، وإما أن يكون للدالة خط تقارب رأسي $x = c$ حيث $c \in [a ، b]$ أو الدالة معرفة على $(a ، b]$ ، $[a ، b)$ (هذا يعني أن الفترة غير محدودة)

ونلخص حالات التكامل المعتل :

1 – إذا كانت أحد حدود التكامل أوكليهما تؤول إلى قيمة لا نهائية

تعريف :

■ إذا كانت f متصلة على الفترة $[a ، \infty)$ فإن :

$$\int_a^{\infty} f(x)\,dx = \lim_{v\to\infty}\int_a^{v} f(x)\,dx$$

■ إذا كانت f متصلة على الفترة $(-\infty، b]$ فإن :

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{u\to\infty}\int_u^{b} f(x)\,dx$$

■ إذا كانت f متصلة على الفترة $(-\infty ، \infty)$ فإن :

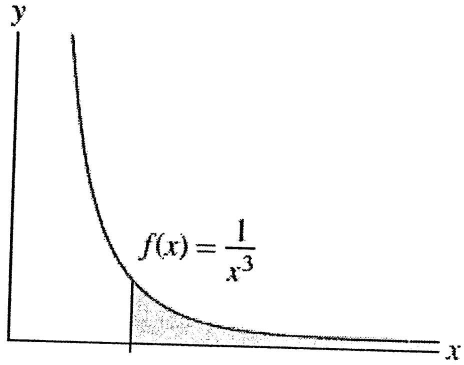
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

$$= \lim_{v\to -\infty}\int_v^{0} f(x)\,dx + \lim_{u\to\infty}\int_0^{u} f(x)\,dx$$

● يكون هذا التكامل المعتل **متقارب** إذا وفقط إذا كانت النهاية موجودة ومحدودة وتكون قيمته في حالة تقاربه مساوية لقيمة النهاية

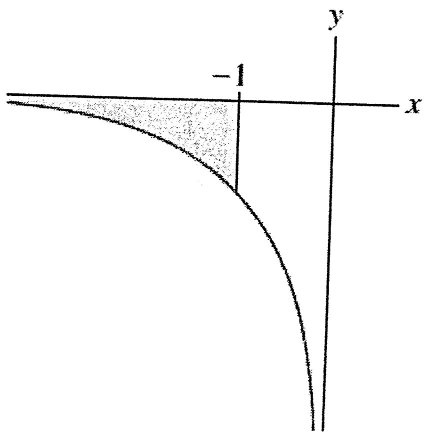
● يكون هذا التكامل المعتل **متباعد** إذا وفقط إذا لم يكن متقارباً أي أنه تكون النهاية إما غير موجوده

مثال /



$$1)\ \int_2^{\infty}\frac{1}{x^3}\,dx=\lim_{v\to\infty}\int_2^{v}\frac{1}{x^3}\,dx=\lim_{v\to\infty}\left[-\frac{1}{2x^2}\right]_2^{v}$$

$$=\lim_{v\to\infty}\left[\frac{1}{8}-\frac{1}{2v^2}\right]=\frac{1}{8}\quad \text{متقارب}$$



$$2)\ \int_{-\infty}^{-1}\frac{1}{x}\,dx=\lim_{u\to-\infty}\int_u^{-1}\frac{1}{x}\,dx=\lim_{u\to-\infty}\left[\ln|u|\right]_u^{-1}$$

$$=\lim_{u\to-\infty}\left[\ln 1-\ln|u|\right]=\lim_{u\to-\infty}\left[-\ln|u|\right]$$

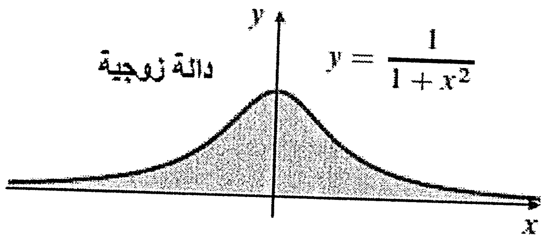
$$=-\lim_{u\to-\infty}\ln|u|=-\infty\quad \text{غير متقارب (متباعد)}$$

$$3)\ \int_0^{\infty}xe^{-x}\,dx=\lim_{v\to\infty}\int_0^{v}xe^{-x}\,dx=\lim_{v\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^{v}$$

$$=\lim_{v\to\infty}\left[-(x+1)e^{-x}\right]_0^{v}=\lim_{v\to\infty}\left[-\frac{x+1}{e^x}\right]_0^{v}=\lim_{v\to\infty}\left[1-\frac{v+1}{e^v}\right]$$

$$=1-\lim_{v\to\infty}\frac{v+1}{e^v}=1-\underbrace{\lim_{v\to\infty}\frac{1}{e^v}}_{\text{لوبيتال}}=1-0=1$$

$$4)\ \int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx=\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx+\int_0^{\infty}\frac{1}{1+x^2}\,dx$$



$$=2\int_0^{\infty}\frac{1}{1+x^2}\,dx=2\lim_{v\to\infty}\int_0^{v}\frac{1}{1+x^2}\,dx$$

$$=2\lim_{v\to\infty}\left[\tan^{-1}x\right]_0^{v}=2\lim_{v\to\infty}\left[\tan^{-1}v-0\right]=2\times\frac{\pi}{2}=\pi$$

$$5)\ \int_1^{\infty}\frac{\ln x}{x^2}\,dx=\lim_{v\to\infty}\int_1^{v}\frac{\ln x}{x^2}\,dx=\lim_{v\to\infty}\left[-\frac{1}{x}\ln x-\frac{1}{x}\right]_1^{v}=\lim_{v\to\infty}\left[-\frac{1}{v}\ln v-\frac{1}{v}+1\right]$$

$$=\lim_{v\to\infty}\left[-\frac{\ln v+1}{v}+1\right]=-\lim_{v\to\infty}\frac{\ln v+1}{v}+\lim_{v\to\infty}1=-\left[\lim_{v\to\infty}\frac{\ln v+1}{v}\right]+1$$

$$=-\left[\lim_{v\to\infty}\frac{\frac{1}{v}}{1}\right]+1=-\left[\lim_{v\to\infty}\frac{1}{v}\right]+1=0+1=1$$

$$6)\ \int_{-\infty}^{\infty}xe^{-x^2}\,dx=\int_{-\infty}^{0}xe^{-x^2}\,dx+\int_0^{\infty}xe^{-x^2}\,dx=\lim_{u\to-\infty}\int_u^{0}xe^{-x^2}\,dx+\lim_{v\to\infty}\int_0^{v}xe^{-x^2}\,dx$$

$$=\lim_{u\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_u^{0}+\lim_{v\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^{v}$$

$$=\frac{1}{2}\lim_{u\to-\infty}\left[e^{-u^2}-1\right]+\frac{1}{2}\lim_{v\to\infty}\left[1-e^{-v^2}\right]$$

$$=\frac{1}{2}\left[\lim_{u\to-\infty}e^{-u^2}-1+1-\lim_{v\to\infty}e^{-v^2}\right]=\frac{1}{2}\left[\lim_{u\to-\infty}e^{-u^2}-\lim_{v\to\infty}e^{-v^2}\right]$$

$$=\frac{1}{2}\left[\lim_{u\to-\infty}\frac{1}{e^{u^2}}-\lim_{v\to\infty}\frac{1}{e^{v^2}}\right]=\frac{1}{2}[1-1]=0$$

$$7)\ \int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dx=\int_{-\infty}^{0}\frac{x}{1+x^2}\,dx+\int_0^{\infty}\frac{x}{1+x^2}\,dx=\lim_{u\to-\infty}\int_u^{0}\frac{x}{1+x^2}\,dx+\lim_{v\to\infty}\int_0^{v}\frac{x}{1+x^2}\,dx$$

$$=\frac{1}{2}\lim_{u\to-\infty}\int_u^{0}\frac{2x}{1+x^2}\,dx+\frac{1}{2}\lim_{v\to\infty}\int_0^{v}\frac{2x}{1+x^2}\,dx$$

$$=\frac{1}{2}\lim_{u\to-\infty}\left[\ln(1+x^2)\right]_u^{0}+\frac{1}{2}\lim_{v\to\infty}\left[\ln(1+x^2)\right]_0^{v}$$

$$=\frac{1}{2}\lim_{u\to-\infty}\left[\ln(1)-\ln(1+u^2)\right]+\frac{1}{2}\lim_{v\to\infty}\left[\ln(1+v^2)-\ln(1)\right]$$

$$=\frac{1}{2}\lim_{u\to-\infty}\left[-\ln(1+u^2)\right]+\frac{1}{2}\lim_{v\to\infty}\left[\ln(1+v^2)\right]$$

$=\infty+\infty=\infty$ متباعد

$$8)\ \int_{-\infty}^{\infty}\frac{x}{1+x^4}\,dx=\int_{-\infty}^{0}\frac{x}{1+x^4}\,dx+\int_{0}^{\infty}\frac{x}{1+x^4}\,dx$$

$$=\frac{1}{2}\lim_{u\to-\infty}\int_{u}^{0}\frac{2x}{1+(x^2)^2}\,dx+\frac{1}{2}\lim_{v\to\infty}\int_{0}^{v}\frac{2x}{1+(x^2)^2}\,dx$$

$$=\frac{1}{2}\lim_{u\to-\infty}\left[\tan^{-1}x^2\right]_u^0+\frac{1}{2}\lim_{v\to\infty}\left[\tan^{-1}x^2\right]_0^v$$

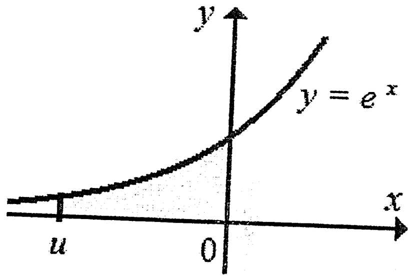
$$=\frac{1}{2}\lim_{u\to-\infty}\left[\tan^{-1}0-\tan^{-1}u^2\right]+\frac{1}{2}\lim_{v\to\infty}\left[\tan^{-1}u^2-\tan^{-1}0\right]$$

$$=\frac{1}{2}\lim_{u\to-\infty}\left[-\tan^{-1}u^2\right]+\frac{1}{2}\lim_{v\to\infty}\left[\tan^{-1}u^2\right]$$

$$=\frac{1}{2}\left(-\frac{\pi}{2}\right)+\frac{1}{2}\left(\frac{\pi}{2}\right)=-\frac{\pi}{4}+\frac{\pi}{4}=0$$

$$9)\ \int_{-\infty}^{-1}\frac{1}{1+x^2}\,dx=\lim_{u\to-\infty}\int_{u}^{-1}\frac{1}{1+x^2}\,dx=\lim_{u\to-\infty}\left[\tan^{-1}x\right]_u^{-1}$$

$$=\lim_{u\to-\infty}\left[\tan^{-1}(-1)-\tan^{-1}u\right]=-\frac{\pi}{4}-(-\frac{\pi}{2})=\frac{\pi}{4}$$



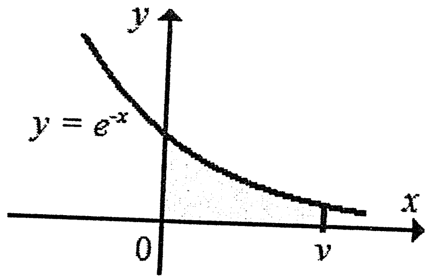
$$10)\ \int_{-\infty}^{\infty}e^{-|x|}\,dx=\int_{-\infty}^{0}e^{x}\,dx+\int_{0}^{\infty}e^{-x}\,dx$$

$$=\lim_{u\to-\infty}\int_{u}^{0}e^{x}\,dx+\lim_{v\to\infty}\int_{0}^{v}e^{-x}\,dx$$

$$=\lim_{u\to-\infty}\left[e^{x}\right]_u^0+\lim_{v\to\infty}\left[e^{-x}\right]_0^v$$

$$=\lim_{u\to-\infty}\left[1-e^{u}\right]+\lim_{v\to\infty}\left[e^{-v}-1\right]$$

$$=1+1=2$$



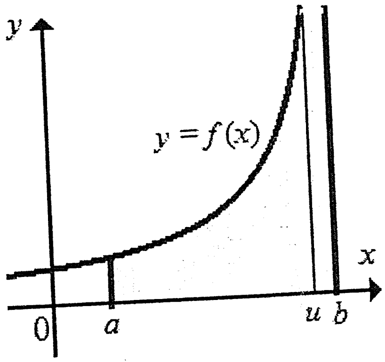
$$11)\ \int_{e}^{\infty}\frac{dx}{x\ln x}=\lim_{v\to\infty}\int_{e}^{v}\frac{dx}{x\ln x}=\lim_{v\to\infty}\left[\ln(\ln x)\right]_e^v=\lim_{v\to\infty}\left[\ln(\ln v)-\ln(\ln e)\right]$$

$$=\lim_{v\to\infty}\left[\ln(\ln v)-\ln 1\right]=\lim_{v\to\infty}\left[\ln(\ln v)\right]=\infty$$

2 – إذا كان $f\left(x\right)$ دالة معرفة على فترة غير محدودة

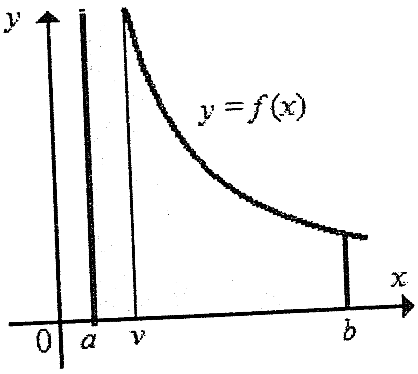
تعريف :

■ إذا كانت f معرفة على الفترة $\left[a \text{ ، } b\right)$ أي أن b نقطة شاذة فإن :



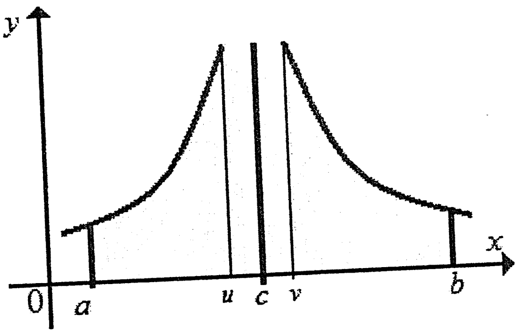
$$\int_a^b f\left(x\right) dx = \lim_{u \to b^-} \int_a^u f\left(x\right) dx$$

■ إذا كانت f متصلة على الفترة $\left(a \text{ ، } b\right]$ أي أن a نقطة شاذة فإن :



$$\int_a^b f\left(x\right) dx = \lim_{v \to a^+} \int_v^b f\left(x\right) dx$$

■ إذا كانت f متصلة على الفترة $\left[a \text{ ، } b\right]$ ، $c \in \left[a \text{ ، } b\right]$ فإن :



$$\int_a^b f\left(x\right) dx = \int_a^c f\left(x\right) dx + \int_c^b f\left(x\right) dx$$

$$= \lim_{u \to c^-} \int_a^u f\left(x\right) dx + \lim_{v \to c^+} \int_v^b f\left(x\right) dx$$

● في أي من الحالات السابقة إذا كانت النهاية موجودة فإن التكامل المعتل **متقارب** وإذا كانت غير موجودة فإن التكامل **متباعد**

12) $\int_0^1 \frac{dx}{x} = \lim_{v \to 0^+} \int_v^1 \frac{1}{x}\, dx = \lim_{v \to 0^+} \left[(\ln x)\right]_v^1 = \lim_{v \to 0^+} \left[\ln 1 - \ln v\right] = \lim_{v \to 0^+} \left[\ln \frac{1}{v}\right] = \infty$

13) $\int_0^1 \frac{dx}{(1-x)^{\frac{1}{3}}} = \lim_{u \to 1^-} \int_1^u (1-x)^{-\frac{1}{3}}\, dx = -\frac{3}{2} \lim_{u \to 1^-} \left[(1-x)^{\frac{2}{3}}\right]_0^u = -\frac{3}{2} \lim_{u \to 1^-} \left[(1-u)^{\frac{2}{3}} - 1\right]$

$= -\frac{3}{2}\left[(1-1)^{\frac{2}{3}} - 1\right] = -\frac{3}{2}(0-1) = \frac{3}{2}$

14) $\int_{-1}^1 \frac{dx}{(x+1)^{\frac{2}{3}}} = \lim_{v \to -1^+} \int_v^1 (x+1)^{-\frac{2}{3}}\, dx = 3 \lim_{v \to -1^+} \left[(x+1)^{\frac{1}{3}}\right]_v^1$

$= 3 \lim_{v \to -1^+} \left[(2)^{\frac{1}{3}} - (v+1)^{\frac{1}{3}}\right] = 3\left[(2)^{\frac{1}{3}} - 0\right] = 3\sqrt[3]{2}$

15) $\int_0^a \frac{dx}{a^2 - x^2} = \int_0^a \frac{dx}{(a-x)(a+x)} = \frac{1}{2a} \int_0^a \left(\frac{1}{a+x} + \frac{1}{a-x}\right) dx = \frac{1}{2a} \lim_{u \to a^-} \left(\frac{1}{a+x} + \frac{1}{a-x}\right) dx$

$= \frac{1}{2a} \lim_{u \to a^-} \left[\ln(a+x) - \ln(a-x)\right]_0^u = \frac{1}{2a} \lim_{u \to a^-} \left[\ln\left(\frac{a+x}{a-x}\right)\right]_0^u$

$= \frac{1}{2a} \lim_{u \to a^-} \left[\ln\left(\frac{a+u}{a-u}\right) - \ln 1\right] = \frac{1}{2a} \lim_{u \to a^-} \left[\ln\left(\frac{a+u}{a-u}\right)\right] = \infty$

16) $\int_0^1 \frac{dx}{x\sqrt{1-x}}$ put $u^2 = 1-x \Rightarrow x = 1-u^2 \Rightarrow dx = -2u\, du$

$x = 0 \Rightarrow u = 1$ ، $x = 1 \Rightarrow u = 0$

$\therefore I = \int_1^0 \frac{-2u\, du}{(1-u^2)u} = -2 \int_1^0 \frac{du}{1-u^2} = 2 \int_0^1 \frac{du}{1-u^2} = \int_0^1 \frac{2}{(1+u)(1-u)} du = \int_0^1 \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du$

$= \lim_{t \to 1^-} \int_0^t \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du = \lim_{t \to 1^-} \left[\ln(1+x) - \ln(1-x)\right]_0^t = \lim_{t \to 1^-} \left[\ln\left(\frac{1+x}{1-x}\right)\right]_0^t$

$= \lim_{t \to 1^-} \left[\ln\left(\frac{1+t}{1-t}\right) - \ln 1\right] = \infty$

17) $\int_0^{\pi/2} \tan x\, dx = \lim_{u \to \frac{\pi}{2}^-} \int_0^u \tan x\, dx = \lim_{u \to \frac{\pi}{2}^-} \left[\ln|\sec x|\right]_0^u = \lim_{u \to \frac{\pi}{2}^-} \left[\ln|\sec u| - \ln|\sec 0|\right]$

$= \lim_{u \to \frac{\pi}{2}^-} \left[\ln|\sec u| - \ln 1\right] = \lim_{u \to \frac{\pi}{2}^-} \left[\ln|\sec u|\right] = \infty$

18) $$\int_1^e \frac{dx}{x\sqrt{\ln x}} = \lim_{v\to 1^+}\int_v^e \frac{1}{x}(\ln x)^{-\frac{1}{2}}\,dx = \lim_{v\to 1^+}\left[2(\ln x)^{\frac{1}{2}}\right]_v^e = 2\lim_{v\to 1^+}\left[(\ln e)^{\frac{1}{2}}-(\ln v)^{\frac{1}{2}}\right]$$

$$= 2\left[(\ln e)^{\frac{1}{2}}-(\ln 1)^{\frac{1}{2}}\right] = 2[1-0] = 2$$

19) $$\int_0^{\pi/2} \frac{\cos x\,dx}{(1-\sin x)^{2/3}} = -\lim_{u\to\frac{\pi}{2}^-}\int_0^u -\cos x\,(1-\sin x)^{-\frac{2}{3}}\,dx = -3\lim_{u\to\frac{\pi}{2}^-}\left[(1-\sin x)^{\frac{1}{3}}\right]_0^u$$

$$= -3\lim_{u\to\frac{\pi}{2}^-}\left[(1-\sin u)^{\frac{1}{3}}-1\right] = -3\left[(1-\sin\tfrac{\pi}{2})^{\frac{1}{3}}-1\right] = -3\left[(0)^{\frac{1}{3}}-1\right] = 3$$

20) $$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^{1/2}\frac{dx}{\sqrt{x-x^2}} + \int_{1/2}^1\frac{dx}{\sqrt{x-x^2}} = 2\int_0^{1/2}\frac{dx}{x-x^2} = 2\int_0^{1/2}\frac{dx}{\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}}$$

$$= \lim_{v\to 0^+} 2\int_0^{1/2}\frac{dx}{\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}} = 2\lim_{v\to 0^+}\left[\sin^{-1}\frac{x-\frac{1}{2}}{\frac{1}{2}}\right]_v^{1/2}$$

$$= 2\lim_{v\to 0^+}\left[\sin^{-1}(2x-1)\right]_v^{1/2} = 2\lim_{v\to 0^+}\left[\sin^{-1}0-\sin^{-1}(2v-1)\right]$$

$$= 2\left[-\sin^{-1}(-1)\right] = 2\left[-(-\frac{\pi}{2})\right] = \pi$$

21) $$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx = \int_2^\infty\left(\frac{2}{x-1}-\frac{2x+1}{x^2+1}\right)dx = \int_2^\infty\left(\frac{2}{x-1}-\frac{2x}{x^2+1}-\frac{1}{x^2+1}\right)dx$$

$$= \lim_{u\to\infty}\int_2^u\left(\frac{2}{x-1}-\frac{2x}{x^2+1}-\frac{1}{x^2+1}\right)dx = \lim_{u\to\infty}\left[2\ln(x-1)-\ln(x^2+1)-\tan^{-1}x\right]_2^u$$

$$= \lim_{u\to\infty}\left[\ln\frac{(x-1)^2}{x^2+1}-\tan^{-1}x\right]_2^u = \lim_{u\to\infty}\left[\left(\ln\frac{(u-1)^2}{u^2+1}-\tan^{-1}u\right)-\left(\ln\frac{1}{5}-\tan^{-1}2\right)\right]$$

$$= \lim_{u\to\infty}\left[\left(\ln\frac{u^2-2u+1}{u^2+1}-\tan^{-1}u\right)-\left(-\ln 5-\tan^{-1}2\right)\right]$$

$$= \lim_{u\to\infty}\left[\left(\ln\frac{1-\frac{2}{u}+\frac{1}{u^2}}{1+\frac{1}{u^2}}-\tan^{-1}u\right)-\left(-\ln 5-\tan^{-1}2\right)\right]$$

$$= \left(0-\frac{\pi}{2}\right)-\left(-\ln 5-\tan^{-1}2\right) = -\frac{\pi}{2}+\ln 5+\tan^{-1}2$$